March 22, 2010

GOOD INDEX BEHAVIOUR OF θ-REPRESENTATIONS, I

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ABSTRACT. Let Q be an algebraic group with $\mathfrak{q} = \mathrm{Lie}\, Q$ and V a Q-module. The index of V is the minimal codimension of the Q-orbits in the dual space V^*. There is a general inequality, due to Vinberg, relating the index of V and the index of a Q_v-module $V/\mathfrak{q}{\cdot}v$ for $v \in V$. A pair (Q, V) is said to have GIB if Vinberg's inequality turns into an equality for all $v \in V$. In this article, we are interested in the GIB property of θ-representations, where θ is a finite order automorphism of a simple Lie algebra $\mathfrak{g}$. An automorphism of order m defines a $\mathbb{Z}/m\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus \mathfrak{g}_i$. If G_0 is the identity component of G^θ, then it acts on $\mathfrak{g}_1$ and this action is called a θ-representation. We classify inner automorphisms of $\mathfrak{gl}_n$ and all finite order autmorphisms of the exceptional Lie algebras such that $(G_0, \mathfrak{g}_1)$ has GIB and $\mathfrak{g}_1$ contains a semisimple element.

1. INTRODUCTION

Let $\mathfrak{q}$ be a Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic zero and V a finite-dimensional $\mathfrak{q}$-module. For $\xi \in V^*$ we set

$$\mathfrak{q}_\xi = \{x \in \mathfrak{q} \mid x{\cdot}\xi = 0\}.$$

Then the non-negative integer

$$\dim V - \max_{\xi \in V^*}(\dim \mathfrak{q}{\cdot}\xi) = \dim V - \dim \mathfrak{q} + \min_{\xi \in V^*}(\dim \mathfrak{q}_\xi)$$

is called the *index* of V; and denoted by $\mathrm{ind}\,(\mathfrak{q}, V)$.

Suppose that $\mathfrak{q}$ is the Lie algebra of an algebraic group Q, and that V is also a Q-module. Then by the Rosenlicht theorem, $\mathrm{ind}\,(\mathfrak{q}, V)$ is equal to $\mathrm{tr.deg}\, \mathbb{F}(V^*)^Q$. Following [13] we say that the pair (Q, V) has a *good index behaviour* (GIB), if

$$\mathrm{ind}\,(\mathfrak{q}, V^*) = \mathrm{ind}\,(\mathfrak{q}_v, (V/\mathfrak{q}{\cdot}v)^*) \tag{1.1}$$

for every $v \in V$. (Note that $V/\mathfrak{q}{\cdot}v$ is a $\mathfrak{q}_v$-module.) It was noticed by Vinberg that the left hand side is less than or equal to the right hand side, see [12, Sect. 1]. Further, (Q, V) is said to have GNIB (Good *Nilpotent* Index Behaviour) if (1.1) holds for all nilpotent elements $v \in V$ (where $v \in V$ is said to be nilpotent if $0 \in \overline{Q{\cdot}v}$).

If $v = 0$, or $\dim(Q{\cdot}v) = \dim V - \mathrm{ind}\,(\mathfrak{q}, V^*)$, or the stabiliser Q_v is reductive, then v satisfies (1.1), see [13]. In general, it is a rather intricate problem to check the equality. One of the possible ways to prove that it holds for v is to find $\xi \in V/\mathfrak{q}{\cdot}v$ such that

$$\dim \mathfrak{q}_v - \dim(\mathfrak{q}_v)_\xi = \dim V - \dim \mathfrak{q}{\cdot}v - \mathrm{ind}\,(\mathfrak{q}, V). \tag{1.2}$$

Note also that if (1.1) is satisfied for v, then it is satisfied for all elements of the orbit $Q{\cdot}v$ as well.

Checking GIB for a representation is even more complicated. No general principle exists at the moment. The only method is to classify the Q-orbits and then compute the index for all of them. A few positive results are known, for example, all representations of an algebraic torus $(\mathbb{F}^{\times})^m$ do have GIB ([13]). It would be interesting to understand what properties of a representation cause GIB to hold.

Suppose that V has only finitely many nilpotent Q-orbits. In that case the representation of Q on V is said to be *observable* and the GIB property is equivalent to GNIB, see [13, Theorem 2.3]. Many observable representations arise in the context of reductive Lie algebras and their semisimple automorphisms.

Let G be a connected reductive algebraic group defined over $\mathbb{F}$ and set $\mathfrak{g} := \operatorname{Lie} G$. The group G acts on $\mathfrak{g}$ via the adjoint representation and this action is known to be observable. In general, if $V = \mathfrak{q}^*$, then $\operatorname{ind}(\mathfrak{q}, \mathfrak{q})$ is the index, $\operatorname{ind}\mathfrak{q}$, of $\mathfrak{q}$ in the sense of Dixmier. If $\gamma \in \mathfrak{q}^*$, then $\mathfrak{q}^*/(\mathfrak{q}{\cdot}\gamma) \cong \mathfrak{q}_\gamma^*$ as a Q_γ-module. Therefore Vinberg's inequality reads $\operatorname{ind}\mathfrak{q} \leqslant \operatorname{ind}\mathfrak{q}_\gamma$. Elashvili conjectured that all reductive Lie algebras have GIB. The conjecture is proved on a case-by-case basis in [5] (exceptional Lie algebras) and [16] (classical Lie algebras). An alternative proof is recently obtained by Charbonnel and Moreau ([2]).

Let θ be an involution of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_i$ is the eigenspace of θ, corresponding to the eigenvalue $(-1)^i$. Here $\mathfrak{g}_0$ is a reductive subalgebra, which is the Lie algebra of the connected reductive subgroup $G_0 \subset G$. The subgroup G_0 acts on $\mathfrak{g}_1$ via restriction of the adjoint representation of G. In many aspects this representation is similar to the adjoint action of a reductive group. For example, all maximal subalgebras in $\mathfrak{g}_1$ consisting of semisimple elements are conjugate under G_0 ([11]). Such a subalgebra $\mathfrak{c} \subset \mathfrak{g}_1$ is usually referred to as a Cartan subspace and $\operatorname{ind}(\mathfrak{g}_0, \mathfrak{g}_1^*) = \dim\mathfrak{c} =: \operatorname{rank}(G_0, \mathfrak{g}_1)$. Kostant and Rallis ([11]) have shown also that the representation of G_0 on $\mathfrak{g}_1$ is observable. As was found out in [13], not all pairs $(G_0, \mathfrak{g}_1)$ satisfy the GIB property. Therefore it is an interesting problem to describe those of them, which do have GIB. In [13], the GIB property was checked for all involutions except the following two (we give the corresponding symmetric pairs $(\mathfrak{g}, \mathfrak{g}_0)$): $(\mathbf{E}_6, \mathfrak{so}_{10}\oplus\mathbb{F})$, $(\mathbf{E}_7, \mathbf{E}_6\oplus\mathbb{F})$. The calculations reported on in this paper show that these two involutions do have GIB.

In the 70-s Vinberg ([15]) generalised results of Kostant and Rallis ([11]) to the set-up of arbitrary semisimple automorphisms of $\mathfrak{g}$. Let θ be an automorphism of $\mathfrak{g}$ of order m and ζ is a primitive m-th root of unity. Then there is a $\mathbb{Z}/m\mathbb{Z}$-grading of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{m-1},$$

where $\mathfrak{g}_i$ is the eigenspace of $\mathfrak{g}$ corresponding to ζ^i. Let $G_0 \subset G$ be a connected algebraic group with the Lie algebra $\mathfrak{g}_0$. Then G_0 is reductive and it acts on $\mathfrak{g}_1$ in a natural way. The group G_0, together with its action on $\mathfrak{g}_1$, is called a θ-group. The representation of G_0 on $\mathfrak{g}_1$ is also called a θ-representation.

Similar to the symmetric space situation a Cartan subspace of $\mathfrak{g}_1$ is defined to be a maximal subspace consisting of commuting semisimple elements. All Cartan subspaces are conjugate under G_0, and the dimension of any of them is called the rank of $\mathfrak{g}_1$ (or rather of the pair $(G_0, \mathfrak{g}_1)$, or of the θ-representation afforded by G_0 and $\mathfrak{g}_1$). According

to [14], [15], all θ-representations are observable. As a consequence, there is always a nilpotent orbit of dimension $\dim \mathfrak{g}_1 - \mathrm{rank}(G_0, \mathfrak{g}_1)$. In [14] Vinberg developed a method for classifying the nilpotent G_0-orbits in $\mathfrak{g}_1$.

In this paper we classify finite order automorphims of the exceptional Lie algebras and inner (finite order) automorphisms of $\mathfrak{gl}_n$ such that $\mathrm{rank}(G_0, \mathfrak{g}_1) > 0$ and $(G_0, \mathfrak{g}_1)$ has GIB. For the exceptional case the answer is given in Tables 1 to 6 in Section 5. In the $\mathfrak{gl}_n$ case a positive rank θ-representation (with inner θ) has GIB if and only if θ is a conjugation by one of the following diagonal matrices:

- $\mathrm{diag}(1^{n-2}, (-1)^2)$, $\mathrm{diag}(1^{n-1}, -1)$, $\mathrm{diag}(1^3, (-1)^3)$ $(n = 6)$, see [13];
- $\mathrm{diag}(1^2, \zeta^2, (\zeta^2)^{n-4})$, where ζ is a primitive 3-d root of unity, see Theorem 4.11;
- $\mathrm{diag}(1, \zeta^{r_1}, \ldots, (\zeta^{m-1})^{r_{m-1}})$, where ζ is a primitive m-th root of unity and there are no subsequences r_i, r_{i+1}, r_{i+2} with all elements being larger than 1, see Theorem 4.14 and Proposition 4.17.

In the $\mathfrak{gl}_n$ case, the classification of nilpotent G_0-orbits, first obtained by Kempken ([10]), is presented in subsection 3.1. In the exceptional case, we get representatives of the nilpotent orbits using the algorithms of [7]. After that GIB is checked for each of them with Algorithm 2.7. The answer, automorphisms θ such that $(G_0, \mathfrak{g}_1)$ has GIB, is given in terms of the so-called *Kac diagrams* ([9]). Here we summarise the main properties that we use of these diagrams; for a more detailed explanation see [8, Chapter X, §5].

A Kac diagram is either an extended Dynkin diagram of $\mathfrak{g}$ or the one obtained from it by gluing together points in the orbits of the diagram automorphism. In addition, one attaches non-negative numbers, labels, to the vertices. We are more interested in the θ-representation, than in the θ itself and there is an easy way to read this from the Kac diagram. Assume for simplicity that θ is inner (in this paper θ is outer only for $\mathfrak{g} = \mathbf{E}_6$). Then G_0 contains a maximal torus of G and the semisimple part of $\mathfrak{g}_0$ is generated by all roots that have labels 0 on the Kac diagram. The lowest weights of $\mathfrak{g}_1$ (with respect to G_0) are in one-to-one correspondence with the roots labeled with 1. Finally, we notice that if $\mathrm{rank}(G_0, \mathfrak{g}_1) > 0$, then the Kac diagram has only labels 0 and 1 ([15]). Hence we give these labels by colouring the nodes: black means that the label is 1, otherwise the label is 0. For a diagram with all nodes black, G_0 is a torus and $(G_0, \mathfrak{g}_1)$ has GIB by [13, Proposition 1.3]. We do not put such "all black" diagrams in the tables.

The GIB property of positive rank automorphisms in other classical types and outer automorphisms of $\mathfrak{gl}_n$ will be studied in a forthcoming paper. Due to the large amount of cases and presumably a rather involved answer (cf. Proposition 4.19) we leave rank zero automorphisms aside. Inner automorphisms in type A provide a remakable instance, where θ-representations are also quiver representations. It would be interesting to check GIB for other (observable) quiver representations.

Acknowledgements. Parts of this work were carried out during our meetings in various mathematical institutions. We are grateful to the University of Trento, Centro Ennio De Giorgi (SNS, Pisa), and Emmy-Noether-Zentrum (Erlangen) for hospitality and financial

support. The second author would like to thank the Max-Planck-Institut für Mathematik (Bonn) for the excellent working conditions it provides.

2. Some remarks on the GIB property

We begin this section with a more explicit description of GIB for θ-representations. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ a $\mathbb{Z}/m\mathbb{Z}$-grading. For $x \in \mathfrak{g}$, let $\mathfrak{g}_x$ denote the centraliser of x in $\mathfrak{g}$ and set $\mathfrak{g}_{i,x} := \mathfrak{g}_x \cap \mathfrak{g}_i$. For $x \in \mathfrak{g}_1$, the inclusion $0 \in \overline{G_0{\cdot}x}$ holds if and only if x is a nilpotent element of $\mathfrak{g}$ in the usual sense, see [15].

Proposition 2.1. (i) *For all nilpotent elements $e \in \mathfrak{g}_1$ we have*

$$\mathrm{rank}(G_0, \mathfrak{g}_1) \leq \mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}).$$

(ii) *The pair $(G_0, \mathfrak{g}_1)$ has GIB if and only if for all nilpotent elements $e \in \mathfrak{g}_1$ we have* $\mathrm{rank}(G_0, \mathfrak{g}_1) = \mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e})$.

Proof. (cf. [13, Proposition 2.6]). In this case (1.1) translates to

$$\mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1^*) = \mathrm{ind}\,(\mathfrak{g}_{0,e}, (\mathfrak{g}_1/[\mathfrak{g}_0, e])^*).$$

Now $\mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1^*)$ is equal to $\dim \mathfrak{g}_1 - \max_{x \in \mathfrak{g}_1} \dim[\mathfrak{g}_0, x]$. By [15, Theorem 5], this is equal to $\mathrm{rank}(G_0, \mathfrak{g}_1)$. The Killing form κ gives a nondegenerate pairing $\kappa : \mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathbb{F}$. Using this pairing we get an isomorphism of $\mathfrak{g}_0$-modules $\mathfrak{g}_1^* \cong \mathfrak{g}_{-1}$. In the same way we get an isomorphism of $\mathfrak{g}_{0,e}$-modules

$$(\mathfrak{g}_1/[\mathfrak{g}_0, e])^* \cong \{y \in \mathfrak{g}_{-1} \mid \kappa(y, [\mathfrak{g}_0, e]) = 0\}.$$

Let y lie in the latter space. Then $0 = \kappa(y, [\mathfrak{g}_0, e]) = \kappa([e, y], \mathfrak{g}_0)$. Now $[e, y] \in \mathfrak{g}_0$ and $\kappa : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{F}$ is nondegenerate. Hence $[e, y] = 0$, and it follows that $\{y \in \mathfrak{g}_{-1} \mid \kappa(y, [\mathfrak{g}_0, e]) = 0\} = \mathfrak{g}_{-1,e}$. Therefore $(\mathfrak{g}_1/[\mathfrak{g}_0, e])^* \cong \mathfrak{g}_{-1,e}$ and Vinberg's inequality turns into $\mathrm{rank}(G_0, \mathfrak{g}_1) \leq \mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e})$. Finally note that the G_0-module $\mathfrak{g}_1$ has only finitely many nilpotent orbits, see [15], and thereby GIB is equivalent to GNIB by [13, Theorem 2.3]. □

We will also need some easy technical statements concerning GIB.

Proposition 2.2. *Suppose that $\mathfrak{g} = \mathrm{Lie}\, G$ is a reductive Lie algebra and V a finite dimensional G-module. Then $\mathrm{ind}\,(\mathfrak{g}, V) = \mathrm{ind}\,(\mathfrak{g}, V^*)$, also V and V^* have (or do not have) GIB at the same time.*

Proof. Each reductive group G posses a so-called *Weyl involution* σ, see e.g. [8, Chapter IX, §5], which has a property that $\rho \circ \sigma \cong \rho^*$ for all finite dimensional representations ρ. In particular, for each $v \in V$, where is a vector $\sigma(v) \in V^*$ such that $G_{\sigma(v)} = \sigma(G_v)$. Since σ preserves the dimension of subgroups, we have $\mathrm{ind}\,(\mathfrak{g}, V) = \mathrm{ind}\,(\mathfrak{g}, V^*)$. Moreover, σ establishes an isomorphism between the representations of G_v on $V/\mathfrak{g}{\cdot}v$ and $G_{\sigma(v)}$ on $V^*/\mathfrak{g}{\cdot}\sigma(v)$. If GIB fails for v, it also fails for $\sigma(v)$. □

Since $\mathfrak{g}_{-1} \cong \mathfrak{g}_1^*$, these θ-representations are of the same rank and GIB holds for $(G_0, \mathfrak{g}_1)$ if and only if it holds for $(G_0, \mathfrak{g}_{-1})$.

Proposition 2.3. *A representation of* $\mathfrak{q}$ *on* V *has GIB if and only if for all* $v \in V$ *there is* $w \in V$ *such that* $\dim(\mathfrak{q}{\cdot}v + \mathfrak{q}_v{\cdot}w) = \dim V - \operatorname{ind}(\mathfrak{q}, V^*)$.

Proof. By definition, the representation has GIB if and only if for all $v \in V$ we have $\operatorname{ind}(\mathfrak{q}_v, (V/\mathfrak{q}{\cdot}v)^*) = \operatorname{ind}(\mathfrak{q}, V^*)$. The equality holds if and only if there is a coset $\bar{w} = w + \mathfrak{q}{\cdot}v$ (with $w \in V$) such that $\dim(V/\mathfrak{q}{\cdot}v) - \dim(\mathfrak{q}_v{\cdot}\bar{w}) = \operatorname{ind}(\mathfrak{q}, V^*)$. It remains to notice that the left hand side is equal to $\dim V - \dim(\mathfrak{q}{\cdot}v) - \dim(\mathfrak{q}_v{\cdot}\bar{w})$ and that $\dim(\mathfrak{q}{\cdot}v) + \dim(\mathfrak{q}_v{\cdot}\bar{w}) = \dim(\mathfrak{q}{\cdot}v + \mathfrak{q}_v{\cdot}w)$. □

Let Q be an algebraic group acting on a finite dimensional vector space V and $\mathfrak{q} = \operatorname{Lie} Q$. Suppose that $\operatorname{ind}(\mathfrak{q}, V^*) = 1$ and $av \notin \mathfrak{q}{\cdot}v$ for generic $v \in V$ and non-zero $a \in \mathbb{F}$. Consider the action of $\widetilde{Q} := Q \times \mathbb{F}^\times$ on V such that $t{\cdot}v = tv$ for all $t \in \mathbb{F}^\times$, $v \in V$. Set $\widetilde{\mathfrak{q}} := \operatorname{Lie} \widetilde{Q}$. The two groups Q and $\widetilde{Q}$ have different generic orbits on V. Hence $\operatorname{ind}(\widetilde{\mathfrak{q}}, V^*) = 0$.

Lemma 2.4. *Suppose that the above assumptions are satisfied and the action of* $\widetilde{Q}$ *on* V *has GIB. Then the action of* Q *on* V *has GIB as well.*

Proof. Take any $v \in V$. Then either $\mathfrak{q}{\cdot}v = \widetilde{\mathfrak{q}}{\cdot}v$ or $\dim \widetilde{\mathfrak{q}}{\cdot}v = \dim \mathfrak{q}{\cdot}v + 1$ and $\mathfrak{q}_v = \widetilde{\mathfrak{q}}_v$. If the first case takes place, then $V/\mathfrak{q}{\cdot}v = V/\widetilde{\mathfrak{q}}{\cdot}v$ and $\dim \mathfrak{q}_v = \dim \widetilde{\mathfrak{q}}_v - 1$. Hence $\operatorname{ind}(\mathfrak{q}_v, (V/\mathfrak{q}{\cdot}v)^*) \leqslant \operatorname{ind}(\widetilde{\mathfrak{q}}_v, (V/\widetilde{\mathfrak{q}}{\cdot}v)^*) + 1 = 1$. In the second case $\mathfrak{q}_v{\cdot}x + \widetilde{\mathfrak{q}}{\cdot}v = V$ for generic $x \in V$. Since $\dim \mathfrak{q}{\cdot}v = \dim \widetilde{\mathfrak{q}}{\cdot}v - 1$, we again get the inequality $\operatorname{ind}(\mathfrak{q}_v, (V/\mathfrak{q}{\cdot}v)^*) \leqslant 1$. □

Remark 2.5. The inverse implication is not true in general.

Let $\mathfrak{q} = \left\{ \begin{pmatrix} -t & s & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} \mid t, s \in \mathbb{F} \right\}$ be a two-dimensional subalgebra of $\mathfrak{gl}_3$. Then $\operatorname{ind}(\mathfrak{q}, (\mathbb{F}^3)^*) = 1$ and the defining representation of $\mathfrak{q}$ on $\mathbb{F}^3$ has GIB. If we add a one dimensional central torus, then $\operatorname{ind}(\widetilde{\mathfrak{q}}, (\mathbb{F}^3)^*) = 0$, but the GIB property is not satisfied for the second basis vector.

One of the ways to compute $\operatorname{ind}(\mathfrak{q}, V)$ is related to the matrix $(\mathfrak{q}{\cdot}V)$ of the action of $\mathfrak{q}$ on V. Let $x_1, \ldots, x_n$ be a basis of $\mathfrak{q}$ and $v_1, \ldots, v_s$ a basis of V. Then $(\mathfrak{q}{\cdot}V)$ is an $n{\times}s$-matrix with entries $x_i{\cdot}v_j$. Each element of V can be considered as a linear (or rational) function on V^*. Therefore it is possible to compute the rank of $(\mathfrak{q}{\cdot}V)$ over a field $\mathbb{F}(V^*)$.

Lemma 2.6. *We have* $\operatorname{ind}(\mathfrak{q}, V) = \dim V - \operatorname{rank}(\mathfrak{q}{\cdot}V)$.

Proof. Take $\xi \in V^*$ and set $c_{ij} := \xi(x_i{\cdot}v_j)$. Let $x = \sum_i \alpha_i x_i \in \mathfrak{q}$ (with $\alpha_i \in \mathbb{F}$). Then

$$x{\cdot}\xi(v_j) = -\xi(x{\cdot}v_j) = -\sum_{i=1}^{n} c_{ij}\alpha_i.$$

Since $x{\cdot}\xi = 0$ if and only if $x{\cdot}\xi(v_j) = 0$ for all $1 \leqslant j \leqslant s$, the stabilisers $\mathfrak{q}_\xi$ consists of all $x = \sum_i \alpha_i x_i$ such that $\sum_{i=1}^n c_{ij}\alpha_i = 0$ for $1 \leqslant j \leqslant s$. Hence $\dim \mathfrak{q}_\xi = \dim \mathfrak{q} - \operatorname{rank} A(\xi)$, where $A(\xi)$ is an $n{\times}s$-matrix with entries c_{ij}. Since $\operatorname{rank} A(\xi) \leqslant \operatorname{rank}(\mathfrak{q}{\cdot}V)$ and the equality holds for generic $\xi \in V^*$, we get $\operatorname{ind}(\mathfrak{q}, V) = \dim V^* - (\dim \mathfrak{q} - (\dim \mathfrak{q} - \operatorname{rank}(\mathfrak{q}{\cdot}V)) = \dim V - \operatorname{rank}(\mathfrak{q}{\cdot}V)$. □

In case $\mathfrak{q} = \mathfrak{g}_{0,e}$, $V = \mathfrak{g}_{-1,e}$, we will denote the matrix $(\mathfrak{q}{\cdot}V)$ by $([\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}])$.

Lemma 2.6 provides an easy method to compute an upper bound for the index that is very likely to be equal to the index. Each s-tuple $\underline{a} = (a_1, \ldots, a_s) \in \mathbb{F}^s$ defines an element $\xi \in V^*$ such that $\xi(v_k) = a_k \in \mathbb{F}$. Set $A(\underline{a}) := A(\xi) = (\xi(x_i{\cdot}v_j))$. Then

$$\mathrm{ind}\,(\mathfrak{q}, V) = \dim V - \max_{\underline{a}\in\mathbb{F}^s} \mathrm{rank}(A(\underline{a})).$$

The entries of $A(\underline{a})$ are linear polynomials in the a_i. It follows that if we take random coefficients a_i then the rank of this matrix is very likely maximal. In other words, for any s-tuple $\underline{a}$ the value of $\dim V - \mathrm{rank} A(\underline{a})$ is an upper bound for $\mathrm{ind}\,(\mathfrak{q}, V)$, and if the a_k are chosen randomly, uniformly, and independently from a large enough set, then, very probably, equal to it.

There are several ways to get the value of the generic rank of $A(\underline{a})$. First of all, we can consider the row space of $A(\underline{a})$ over the ground field $\mathbb{F}$, where we consider the a_i as linearly independent indeterminates. We can replace the rows by an $\mathbb{F}$-linearly independent set of rows that span the same space over $\mathbb{F}$. We can do the same with the columns. Denote the resulting matrix by $\widetilde{A}(\underline{a})$. Then the generic ranks of $A(\underline{a})$ and $\widetilde{A}(\underline{a})$ are the same. If the lower bound that we get for the rank by substituting a point $\underline{a}$ is equal to the number of columns, or rows, of $\widetilde{A}(\underline{a})$, then we know that this lower bound is the correct value of the generic rank. Otherwise we can compute the rank of $\widetilde{A}(\underline{a})$, where the a_i are indeterminates of a function field over $\mathbb{F}$. We do remark, however, that this operation can be computationally expensive.

On the basis of Proposition 2.1 we formulate the following algorithm.

Algorithm 2.7. Input: a nilpotent elements $e \in \mathfrak{g}_1$ and $\mathrm{rank}(G_0, \mathfrak{g}_1)$.
Output: TRUE if $\mathrm{rank}(G_0, \mathfrak{g}_1) = \mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e})$, FALSE otherwise.

(1) By linear algebra we compute bases of $\mathfrak{g}_{0,e}$ and $\mathfrak{g}_{-1,e}$.
(2) We compute the matrix $A(\underline{a})$ corresponding to the $\mathfrak{g}_{0,e}$-module $\mathfrak{g}_{-1,e}$.
(3) By trying random values for the a_i find a lower bound r for the generic rank of $A(\underline{a})$.
(4) If $\dim \mathfrak{g}_{-1,e} - r = \mathrm{rank}(G_0, \mathfrak{g}_1)$ then output TRUE, else execute the next step.
(5) Compute $\widetilde{A}(\underline{a})$; if r is equal to the number of columns, or rows, of this matrix, then output FALSE, else execute the next step.
(6) Let r' be the rank of $\widetilde{A}(\underline{a})$, where the a_i are indeterminates of a function field over $\mathbb{F}$; if $\dim \mathfrak{g}_{-1,e} - r' = \mathrm{rank}(G_0, \mathfrak{g}_1)$ then output TRUE, else output FALSE.

Lemma 2.8. *The previous algorithm is correct.*

Proof. We note that $\dim \mathfrak{g}_{-1,e} - r$ is an upper bound for $\mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e})$. By Proposition 2.1, $\mathrm{rank}(G_0, \mathfrak{g}_1)$ is a lower bound for this index. So if they are equal, then we know that we have the correct index. If these are not equal, and r is equal to the number of columns, or rows, of $\widetilde{A}(\underline{a})$, then the index is strictly bigger than $\mathrm{rank}(G_0, \mathfrak{g}_1)$. Finally, if this also does not hold, then the last "brute force" step gives the correct value. □

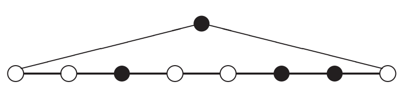
In order to check GIB for $(G_0, \mathfrak{g}_1)$, we can do the following. First we compute representatives of the nilpotent G_0-orbits in $\mathfrak{g}_1$, using the algorithms of [7]. Since $\mathrm{rank}(G_0, \mathfrak{g}_1) = \dim \mathfrak{g}_1 - \max_{e\in\mathfrak{g}_1} \dim(G_0{\cdot}e)$, where e is a nilpotent element, these calculations also provide the value of $\mathrm{rank}(G_0, \mathfrak{g}_1)$. Then for each representative e of a nilpotent G_0-orbit in $\mathfrak{g}_1$ we execute Algorithm 2.7. If the output is TRUE for all representatives of the nilpotent G_0-orbits in $\mathfrak{g}_1$, then $(G_0, \mathfrak{g}_1)$ has GIB by Lemma 2.8 and Proposition 2.1. If FALSE is returned once, then $(G_0, \mathfrak{g}_1)$ does not have GIB.

Remark 2.9. In practice it is a good idea to delay the execution of the expensive Step 6 of Algorithm 2.7. First one collects all nilpotent orbits for which the random procedure indicates that GIB fails. Then Step 6 is executed only once, on the smallest matrix $\widetilde{A}(\underline{a})$. If the corresponding orbit does have GIB, which is very unlikely, or the matrix is too complicated for computer calculations, one may look on other suspicious orbits.

3. Basic facts concerning semisimple inner automorphisms in type A

Let $\mathbb{V}$ be a finite dimensional vector space over $\mathbb{F}$, $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$, and θ an inner automorphism of $\mathfrak{g}$ of order m. We consider θ as an element of the group $G = \mathrm{GL}(\mathbb{V})$ acting on $\mathfrak{g}$ by conjugation. Let ζ be a primitive m-th root of unity. Set $\mathbb{V}_t := \{v \in \mathbb{V} \mid \theta(v) = \zeta^t v\}$ and $r_t = \dim \mathbb{V}_t$. Up to a G-conjugation, θ is uniquely defined by the multiplicities vector $\hat{r} := (r_0, r_1, \ldots, r_{m-1})$. Cyclic permutations of the entries of $\hat{r}$ correspond to multiplications by central elements of $\mathrm{GL}(\mathbb{V})$ and the resulting vectors $\hat{r}'$ define the same automorphism. We have $G_0 = \mathrm{GL}(\mathbb{V}_0)\times \ldots \times\mathrm{GL}(\mathbb{V}_{m-1})$ and $\mathfrak{g}_1 \cong \bigoplus_{i=0}^{m-1} \mathrm{Hom}(\mathbb{V}_i, \mathbb{V}_{i+1})$, where $i+1$ is considered modulo m. Having $\hat{r}$, it is possible to write a Kac diagram of the corresponding θ and vice versa. Since we are not going to use this correspondence, it is only illustrated on one example.

Example 3.1. Let θ be an automorphism of $\mathfrak{gl}_9$ with the Kac diagram:



Then θ is defined by $\hat{r} = (3, 3, 1, 2)$.

Proposition 3.2. ([15, § 3]) *If θ is defined by $\hat{r}$, then* $\mathrm{rank}(G_0, \mathfrak{g}_1) = \min\limits_{i=0,\ldots,m-1} r_i$.

Suppose that $x \in \mathfrak{g}_1$. Let $x = x_s + x_n$ be the Jordan decomposition of x in $\mathfrak{g}$. Due to its uniqueness, we have $x_s, x_n \in \mathfrak{g}_1$. In other words, $\mathfrak{g}_1$ inherits the Jordan decomposition from $\mathfrak{g}$. The rôle of semisimple and nilpotent elements in checking GIB is explained in [13, Section 2]. Suppose that $s \in \mathfrak{g}_1$ is a semisimple element. Then the action of $G_{0,s}$ on $\mathfrak{g}_1/[\mathfrak{g}_0, s]$ is called a *slice* representation of $(G_0, \mathfrak{g}_1)$. By the same argument as in the proof of Proposition 2.1, $(\mathfrak{g}_1/[\mathfrak{g}_0, s])^* \cong \mathfrak{g}_{-1,s}$.

Lemma 3.3. *Suppose that θ is defined by a vector $\hat{r} = (r_0, r_1, \ldots, r_{m-1})$ and the corresponding representation of G_0 on $\mathfrak{g}_1$ has GIB. Let $b \in \mathbb{N}$ be such that $b \leqslant \mathrm{rank}(G_0, \mathfrak{g}_1)$. Then the θ-representation corresponding to $\hat{r}' = (r_0 - b, r_1 - b, \ldots, r_{m-1} - b)$ has GIB.*

Proof. By inductive reasons, it suffices to prove the lemma for $b = 1$. Thus assume that $\mathrm{rank}(G_0, \mathfrak{g}_1) > 0$ and $b = 1$. Let $s \in \mathfrak{g}_1$ be a non-zero semisimple element. Set $r_i' := r_i - 1$. Then the slice representation of $G_{0,s}$ on $\mathfrak{g}^*_{-1,s}$ is equivalent to the representation of

$$\mathbb{F}^\times \times \mathrm{GL}_{r_0'} \times \mathrm{GL}_{r_1'} \times \ldots \times \mathrm{GL}_{r_{m-1}'} \ \text{ on } \ \mathbb{F} \oplus \left(\bigoplus_{i=0}^{m-1} (\mathbb{F}^{r_i'})^* \otimes \mathbb{F}^{r_{i+1}'} \right),$$

where the first subgroup $\mathbb{F}^\times$ acts on $\mathfrak{g}^*_{-1,s}$ trivially and the first subspace $\mathbb{F}$ is a trivial $G_{0,s}$-module. It follows that $(G_{0,s}, \mathfrak{g}^*_{-1,s})$ has GIB if and only if the θ-representation corresponding to $\hat{r}'$ has GIB. It remains to notice that GIB of $(G_0, \mathfrak{g}_1)$ implies GNIB of $(G_{0,s}, \mathfrak{g}^*_{-1,s})$ by [13, Theorem 2.1] and in our setting GNIB is equivalent to GIB by [13, Theorem 2.3]. □

3.1. **Nilpotent orbits.** Nilpotent elements in $\mathfrak{gl}(\mathbb{V})$ are classified in terms of their Jordan normal form. A similar classification is possible for nilpotent G_0-orbits in $\mathfrak{g}_1$. This problem was solved by Kempken in [10, §2.II]. Below we present her answer in a slightly modified form.

Let $e \in \mathfrak{g}_1$ be a nilpotent element represented by a partition $(d_1{+}1, \ldots, d_k{+}1)$ of $\dim \mathbb{V}$. Since the subspace $e{\cdot}\mathbb{V} \subset \mathbb{V}$ is θ-invariant and θ acts on $\mathbb{V}$ as a semisimple element, there is a θ-invariant complement to $e{\cdot}\mathbb{V}$, let us say, $\mathbb{W}$. Clearly $\dim \mathbb{W} = k$. Assume that $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_k$. For each number s the subspaces $\ker e^s = \{v \in \mathbb{V} \mid e^s{\cdot}v = 0\}$ and $\ker e^s \cap \mathbb{W}$ are both θ-invariant. Hence a generator w_1 of the maximal Jordan block, i.e., a vector such that $e^{d_1}{\cdot}w_1 \neq 0$, can be chosen as an eigenvector of θ. Set $\mathbb{V}[1] := \mathrm{span}\{w_1, e{\cdot}w_1, \ldots, e^{d_1}{\cdot}w_1\}$. Let $\mathbb{W}'$ be a θ-invariant complement of $\mathbb{F}w_1$ in $\mathbb{W}$. Then also $\mathbb{V} = \mathbb{V}[1] \oplus (\mathbb{W}' \oplus e{\cdot}\mathbb{W}')$ is a θ-invariant decomposition. Proceeding by induction on the number of Jordan blocks we prove that all generators w_i can be chosen as eigenvectors of θ, i.e., $\mathbb{W}$ has a basis $w_1, w_2, \ldots, w_k$ consisting of θ-eigenvectors, where in addition the vectors $e^j{\cdot}w_i$ with $1 \leqslant i \leqslant k$, $0 \leqslant j \leqslant d_i$ form a basis of $\mathbb{V}$. Note that $e^{d_i+1}{\cdot}w_i = 0$ for all $i \leqslant k$. If $\theta(w_i) = \zeta^{t(i)} w_i$, then $\theta(e^s{\cdot}w_i) = \theta(e^s){\cdot}\theta(w_i) = \zeta^{t(i)+s} e^s{\cdot}w_i$. In particular, $\theta(e{\cdot}w_i) = \zeta^{t(i)+1} e{\cdot}w_i$. Summing up:

> to each nilpotent element $e \in \mathfrak{g}_1$ we associate its partition and the θ-eigenvalues $\zeta^{t(i)}$ on the Jordan blocks' generators w_i.

In order to see what nilpotent elements do appear in $\mathfrak{g}_1$, we have to take a partition, choose θ-eigenvalues for the generators w_i and count the dimensions of the eigenspaces according to the rule $\theta(e^s{\cdot}w_i) = \zeta^{t(i)+s} e^s{\cdot}w_i$. If they coincide with the r_t's, then e lies in $\mathfrak{g}_1$.

3.2. **Basis of a centraliser.** In order to do explicit calculations, one needs bases in $\mathfrak{g}_{0,e}$ and $\mathfrak{g}_{-1,e}$. First we introduce a basis in $\mathfrak{g}_e$. If $\xi \in \mathfrak{g}_e$, then $\xi{\cdot}(e^j{\cdot}w_i) = e^j{\cdot}(\xi{\cdot}w_i)$, hence ξ is completely determined by its values on $\mathbb{W}$. The only restriction on $\xi{\cdot}w_i$ is that $e^{d_i+1}{\cdot}(\xi{\cdot}w_i) = \xi{\cdot}(e^{d_i+1}{\cdot}w_i) = 0$. Since vectors $e^s{\cdot}w_i$ form a basis of $\mathbb{V}$, the centraliser $\mathfrak{g}_e$ has a basis $\{\xi_i^{j,s}\}$ such that

$$\left\{ \begin{array}{l} \xi_i^{j,s}{\cdot}w_i = e^s{\cdot}w_j, \\ \xi_i^{j,s}{\cdot}w_t = 0 \ \text{ for } t \neq i, \end{array} \right. \qquad 1 \leqslant i, j \leqslant k, \ \text{ and } \ \max\{d_j - d_i, 0\} \leqslant s \leqslant d_j\ .$$

It is convenient to assume that $\xi_i^{j,s} = 0$ whenever s does not satisfy the above restrictions. The composition rule shows that the basis elements $\xi_i^{j,s}$ satisfy the following commutator relation:

$$[\xi_i^{j,s}, \xi_p^{q,t}] = \delta_{q,i}\xi_p^{j,t+s} - \delta_{j,p}\xi_i^{q,s+t}, \tag{3.1}$$

where $\delta_{i,j} = 1$ if $i = j$ and is zero otherwise. Each $\xi_i^{j,s}$ is an eigenvector of θ with

$$\theta(\xi_i^{j,s}) = \zeta^{s+t(j)-t(i)}\xi_i^{j,s}. \tag{3.2}$$

This allows one to compute $\mathfrak{g}_{0,e}$ and $\mathfrak{g}_{-1,e}$.

Example 3.4. Let θ be an automorphism of $\mathfrak{gl}_9$ with $\hat{r} = (3,3,3)$. Then there is a nilpotent element $e \in \mathfrak{g}_1$ defined by a partition $(5,3,1)$ such that $\theta(w_1) = w_1$, $\theta(w_2) = \zeta w_2$, and $\theta(w_3) = \zeta^2 w_3$. Indeed, let us put the eigenvalues of θ, or rather the corresponding exponents of ζ, in the squares of the Young diagram corresponding to e.

$$\begin{array}{|c|cc}
\hline
1 & & \\
\hline
0 & & \\
\hline
2 & \multicolumn{1}{|c|}{0} & \\
\hline
1 & \multicolumn{1}{|c|}{2} & \\
\hline
0 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\
\hline
\end{array}$$

Then one readily sees that there are three 0, three 1, and three 2 in the figure. The centraliser $\mathfrak{g}_e$ has a basis

$$\xi_1^{1,0}, \xi_1^{1,2}, \xi_1^{1,2}, \xi_1^{1,3}, \xi_1^{1,4},\ \ \xi_1^{2,0}, \xi_1^{2,1}, \xi_1^{2,2},\ \ \xi_1^{3,0},\ \ \xi_2^{2,0}, \xi_2^{2,1}, \xi_2^{2,2},\ \ \xi_2^{1,2}, \xi_2^{1,3}, \xi_2^{1,4},\ \ \xi_2^{3,0},\ \ \xi_3^{3,0}, \xi_3^{1,4}, \xi_3^{2,2},$$

where the θ-eigenvalues are

$$1, \zeta, \zeta^2, 1, \zeta,\ \ \zeta, \zeta^2, 1,\ \ \zeta^2,\ 1, \zeta, \zeta^2,\ \ \zeta, \zeta^2, 1,\ \ \ \zeta, 1, \zeta^2, \zeta,$$

respectively. In particular $\dim \mathfrak{g}_{0,e} = 6$, $\dim \mathfrak{g}_{1,e} = 7$, and $\dim \mathfrak{g}_{-1,e} = 6$.

4. GIB IN TYPE A

In this section we consider inner finite order automorphisms θ of $\mathfrak{gl}_n$. All θ such that $\operatorname{rank}(G_0, \mathfrak{g}_1) > 0$ and the pair $(G_0, \mathfrak{g}_1)$ has GIB are classified. According to [13], there are only three such involutions, namely $(\mathfrak{g}, \mathfrak{g}_0)$ must be one of the following pairs: $(\mathfrak{gl}_{n+2}, \mathfrak{gl}_n \oplus \mathfrak{gl}_2)$, $(\mathfrak{gl}_{n+1}, \mathfrak{gl}_n \oplus \mathfrak{gl}_1)$, $(\mathfrak{gl}_6, \mathfrak{gl}_3 \oplus \mathfrak{gl}_3)$. Not surprisingly, for automorphisms of higher orders the GIB property can be satisfied only if $\operatorname{rank}(G_0, \mathfrak{g}_1) \leqslant 2$. All initial, so to say, nilpotent orbits without GIB were found on computer. After that it is possible to extend these bad examples to higher dimensions. We also check on computer that some θ-representations of small dimension do have GIB. The computer calculations were done using our implementation of Algorithm 2.7.

The difference between θ-representations of $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ is almost neglectable. Sometimes the general linear algebra is more convenient for calculations. On the other hand, we always have to take into account the central torus, which acts on $\mathfrak{g}$ and $\mathfrak{g}_1$ trivially.

From now on let z be a central element in $\mathfrak{gl}_n$. We deal with automorphisms θ in terms of the corresponding vectors $\hat{r}$, as defined in Section 3.

Although the goal is to classify automorphisms of positive rank having GIB, we first give an example of a θ-representation with $\operatorname{rank}(G_0, \mathfrak{g}_1) = 0$.

Example 4.1. Suppose that $m = 3$ and θ has rank zero, i.e., $\hat{r} = (a, b, 0)$ up to a cyclic permutation. Then $(G_0, \mathfrak{g}_1)$ has GIB. Indeed, $\mathfrak{g}_1 = M_{a,b}(\mathbb{F})$ and G_0-orbits $G_0{\cdot}x \subset \mathfrak{g}_1$ are classified by the rank p of an $a{\times}b$-matrix x. The quotient space $\mathfrak{g}_1/[\mathfrak{g}_0, x]$ is isomorphic to $M_{a-p,b-p}(\mathbb{F})$ and $G_{0,x}$ acts on it as $\mathrm{GL}_{a-p}{\times}\mathrm{GL}_{b-p}$, hence, with an open orbit.

The θ-representation of Example 4.1 appears as a slice representation for the action of G_0 on $\mathfrak{g}_1$ corresponding to $\hat{r} = (a+1, b+1, 1)$. This second θ-representation has GIB as well. In order to prove it, we need three following lemmas.

Lemma 4.2. *Suppose that $\hat{r} = (1, n, 1, 0)$. Then the corresponding representation of G_0 on $\mathfrak{g}_1$ has GIB.*

Proof. Here the action of G_0 on $\mathfrak{g}_1 = \mathbb{F}^n \oplus (\mathbb{F}^n)^*$ has a one-dimensional ineffective kernel, say Q_0, and $H := G_0/Q_0 = \mathbb{F}^{\times}{\times}\mathrm{SL}_n{\times}\mathbb{F}^{\times}$. We have $(t_1, t_2){\cdot}(v_1 + v_2) = t_1 v_1 + t_2 v_2$ for $(t_1, t_2) \in \mathbb{F}^{\times}{\times}\,\mathbb{F}^{\times}$, $v_1 \in \mathbb{F}^n$, $v_2 \in (\mathbb{F}^n)^*$. Set $\mathfrak{h} = \operatorname{Lie} H$. According to Proposition 2.3, we have to show that for all $v = v_1 + v_2 \in \mathfrak{g}_1$, there is $w \in \mathfrak{g}_1$ such that $\mathfrak{h}_v{\cdot}w + \mathfrak{h}{\cdot}v = \mathfrak{g}_1$. In cases $v = 0$, where $\mathfrak{h}_v{\cdot}w = \mathfrak{g}_1$ for generic $w \in \mathfrak{g}_1$; and $v_2(v_1) \neq 0$, where $\mathfrak{h}{\cdot}v = \mathfrak{g}_1$, the statement is clear. If one of the vectors v_1, v_2 is zero, without loss of generality we may assume that $v_2 = 0$, and the other one is not (now $v_1 \neq 0$), then $\mathfrak{h}{\cdot}v = \mathbb{F}^n$ and $\mathfrak{h}_v{\cdot}w_2 = (\mathbb{F}^n)^*$ for generic $w_2 \in (\mathbb{F}^n)^*$.

It remains to treat the case where v_1, v_2 are both non-zero, but $v_2(v_1) = 0$. This implies that $n \geqslant 2$. Here $\dim(\mathfrak{h}{\cdot}v) = 2n-1$ and if $w_2 \in (\mathbb{F}^n)$ is such that $w_2(v_1)$, then $\mathbb{F}w_2 \cap \mathfrak{h}{\cdot}v = \{0\}$. Let $\rho(\mathbb{F}^{\times}) \in \mathrm{SL}_n$ be a one-dimensional torus such that $\rho(t){\cdot}(v_1 + v_2) = tv_1 + tv_2$. Take an element $g_t = (t^{-1}, \rho(t), t^{-1}) \in H$. Then $g_t{\cdot}w_2 \in t^{-2}w_2 + \mathfrak{h}{\cdot}v$ and therefore $\mathfrak{h}_v{\cdot}w_2$ contains $\mathbb{F}w_2$. We conclude that $\mathfrak{h}_v{\cdot}w_2 + \mathfrak{h}{\cdot}v = \mathfrak{g}_1$. □

Lemma 4.3. *Suppose that $\hat{r} = (1, a, b, 1, 0)$. Then the corresponding representation of G_0 on $\mathfrak{g}_1$ has GIB.*

Proof. Using explicit matrix calculations we check that all orbits in $\mathfrak{g}_1$ satisfy the GIB property. Here $G_0 = \mathbb{F}^{\times}{\times}\mathrm{GL}_a{\times}\mathrm{GL}_b{\times}\mathbb{F}^{\times}$ and there is a G_0-invariant decomposition $\mathfrak{g}_1 = V_1 \oplus V_2 \oplus V_3$, where $V_1 \cong (\mathbb{F}^a)^*$, $V_2 \cong \mathbb{F}^a{\otimes}(\mathbb{F}^b)^* \cong M_{a,b}(\mathbb{F})$, and $V_3 \cong \mathbb{F}^b$. Take $e \in \mathfrak{g}_1$ and let $x \in M_{a,b}(\mathbb{F})$ be its projection on V_2 (this means that $e \in x + V_1 + V_3$). Let q be the rank of the matrix x. Replacing e by another element in $G_0{\cdot}e$ we may (and will) assume that x is an identity $q{\times}q$ matrix standing in the upper left corner. Then the stabiliser $G_{0,x}$ is a product $\mathbb{F}^{\times}{\times}\mathrm{GL}_{a-q}{\times}U_a{\times}\mathrm{GL}_q{\times}U_b{\times}\mathrm{GL}_{b-q}{\times}\mathbb{F}^{\times}$, where GL_q is embedded diagonally into $\mathrm{GL}_a{\times}\mathrm{GL}_b$ and U_a, U_b are unipotent radicals of standard parabolics in $\mathrm{GL}_a, \mathrm{GL}_b$, respectively. Set $\mathfrak{u}_a = \operatorname{Lie} U_a$, $\mathfrak{u}_b = \operatorname{Lie} U_b$.

Now $V_1 = (\mathbb{F}^q)^* \oplus (\mathbb{F}^{a-q})^*$, $V_3 = \mathbb{F}^q \oplus \mathbb{F}^{b-q}$, where GL_q acts non-trivially only on $\mathbb{F}^q \oplus (\mathbb{F}^q)^*$, the subgroup GL_{a-q} only on $(\mathbb{F}^{a-q})^*$, and GL_{b-q} only on $\mathbb{F}^{b-q}$. For the nilpotent radicals

we have $[\mathfrak{u}_a, (\mathbb{F}^q)^*] = (\mathbb{F}^{a-q})^*$, if $q \neq 0$, while $[\mathfrak{u}_a, (\mathbb{F}^{a-q})^*] = 0$ and likewise $[\mathfrak{u}_b, \mathbb{F}^q] = \mathbb{F}^{b-q}$, if $q \neq 0$, while $[\mathfrak{u}_b, \mathbb{F}^{b-q}] = 0$. According to this decomposition, we write e is a sum of five vectors $e = v_1 + v_1' + x + v_3 + v_3'$ with $v_1 \in (\mathbb{F}^q)^*$, $v_3 \in \mathbb{F}^q$. Set $W = \mathbb{F}^q \oplus (\mathbb{F}^q)^*$. The reductive part of $G_{0,x}$ acts on W in exactly the same way as the θ-group in Lemma 4.2. Let H be its image in $\mathrm{GL}(W)$ and $\mathfrak{h} := \mathrm{Lie}\, H$. Then for some vector $w \in W$, and therefore for all vectors of an open subset, holds $\mathfrak{h}_{v_1+v_3}{\cdot}w + \mathfrak{h}{\cdot}(v_1{+}v_3) = W$. There are three different possibilities, which are treated separately.

1. Suppose that v_1 and v_3 are both non-zero. Replacing, if necessary, e by an element in $(U_a{\times}U_b){\cdot}e$, we may assume that $v_1' = v_3' = 0$. Let $w \in W$ be a generic vector and $y \in V_2$ a matrix such that its lower right $(a{-}q){\times}(b{-}q)$ submatrix is of the full rank $\min(a{-}q, b{-}q)$. Then

$$
\begin{aligned}
&[\mathfrak{g}_{0,e}, w+y] + [\mathfrak{g}_0, e] = [\mathfrak{g}_{0,e}, w+y] + [\mathfrak{g}_{0,x}, v_1+v_3] + [\mathfrak{g}, e] = \\
&[\mathfrak{gl}_{a-q}{\oplus}\mathfrak{gl}_{b-q}, y] + \mathfrak{h}_{v_1+v_3}{\cdot}w + \mathfrak{h}{\cdot}(v_1{+}v_3) + [\mathfrak{u}_a{\oplus}\mathfrak{u}_b, v_1+v_3] + [\mathfrak{g}_0, e] = \\
&= W + (\mathbb{F}^{q-a})^* + \mathbb{F}^{q-b} + [\mathfrak{gl}_{a-q}{\oplus}\mathfrak{gl}_{b-q}, y] + [\mathfrak{g}_0, x] = \mathfrak{g}_1.
\end{aligned}
\tag{4.1}
$$

By Proposition 2.3, the element e has GIB.

2. Suppose now that one of the vectors v_1, v_3 is zero, but the other one is not. Without loss of generality we may assume that $v_1 \neq 0$, $v_3 = 0$. Now v_3' cannot be assumed to be zero, but $\mathfrak{u}_b \subset \mathfrak{g}_{0,e}$. In the equation (4.1) we have to replace $\mathfrak{gl}_{b-q}$ by $(\mathfrak{gl}_{b-q})_{v_3'}$ in $[\mathfrak{gl}_{a-q}{\oplus}\mathfrak{gl}_{b-q}, y]$; and $[\mathfrak{u}_b, v_3]$ by $[\mathfrak{u}_b, w]$, which is equal to $\mathbb{F}^{b-q}$ for generic w and is a subset of $[\mathfrak{g}_{0,e}, w+y]$. Finally notice that still $[\mathfrak{gl}_{a-q}{\oplus}(\mathfrak{gl}_{b-q})_{v_3'}, y] + [\mathfrak{g}_0, x] = V_2$. Therefore $[\mathfrak{g}_{0,e}, w+y] + [\mathfrak{g}_0, e] = \mathfrak{g}_1$.

3. The last possibility is that $v_1 = v_2 = 0$. If $v_1' \neq 0$ and $v_3' \neq 0$, then $[(\mathfrak{gl}_{a-q})_{v_1'}{\oplus}(\mathfrak{gl}_{b-q})_{v_3'}, y] + [\mathfrak{g}_0, x]$ is a subspace of codimension 1 in V_2. Otherwise the sum is the whole of V_2 and again $\mathfrak{g}_1 = [\mathfrak{g}_{0,e}, w+y] + [\mathfrak{g}_0, e]$. Thus we may safely assume that both vectors are non-zero. In particular, $V_1{\oplus}V_3 \subset [\mathfrak{g}_0, e] + V_2$.

We have $\mathfrak{g}_1/[\mathfrak{g}_0, e] = \mathfrak{w}_1{\oplus}\mathfrak{w}_2{\oplus}\mathfrak{w}_3$, where $\mathfrak{w}_2 \subset V_2$, $\mathfrak{w}_2 \cong M_{a-q,b-q}(\mathbb{F})$, $\mathfrak{w}_1 \cong (\mathbb{F}^q)^*$ is embedded anti-diagonally into $V_1{\oplus}V_2$, and $\mathfrak{w}_1 \cong \mathbb{F}^q$ is embedded anti-diagonally into $V_2{\oplus}V_3$. Next

$$G_{0,e} = \mathrm{GL}_q \times U_a \times U_b \times (\mathbb{F}^{\times}{\times}\mathrm{GL}_{a-q}{\times}\mathrm{GL}_{b-q}{\times}\mathbb{F}^{\times})_{(v_1'+v_2')}.$$

Let $y \in \mathfrak{w}_2$ be a matrix of the maximal rank. Then $\mathfrak{u}_a{\cdot}y = \mathfrak{w}_1$, $\mathfrak{u}_b{\cdot}y = \mathfrak{w}_3$, and $\mathfrak{g}_{0,e}{\cdot}y = \mathfrak{g}_1/[\mathfrak{g}_0, e]$. Thereby $\mathrm{ind}\,(\mathfrak{g}_{0,e}, (\mathfrak{g}_1/[\mathfrak{g}_0, e])^*) = 0$. □

Lemma 4.4. *Let θ be an automorphism of a Lie algebra $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$ of order m defined by a vector $\hat{r} = (1, r_1, \ldots, r_{m-1})$. Suppose that θ' is an automorphism of order $m+1$ of another Lie algebra $\mathfrak{h} = \mathfrak{gl}(\mathbb{V}')$, where $\dim \mathbb{V}' = \dim \mathbb{V} + 1$, defined by a vector $\hat{r}' = (1, r_1, \ldots, r_{m-1}, 1, 0)$, and θ' has GIB. Then θ has GIB as well.*

Proof. Note that $\mathfrak{g}_1 = \mathfrak{h}_1$. The group G_0 is smaller than H_0 and does not contain the central torus of H_0 (acting on $\mathfrak{g}_1$ by the scalar multiplications). Therefore we are in the setting of Lemma 2.4 and the result follows from it. □

In case $m = 3$ we can get a complete answer. This is achieved in a few following steps. We will need a machinery developed in subsections 3.1 and 3.2.

Proposition 4.5. *Suppose that $m = 3$ and θ has rank one, i.e., $\hat{r} = (a, b, 1)$ with $a, b > 0$. Then $(G_0, \mathfrak{g}_1)$ has GIB.*

Proof. We get the result combining Lemmas 4.3 and 4.4. □

Lemma 4.6. *Suppose that $\hat{r} = (2, 2, a)$ with $0 \leqslant a \leqslant 4$. Then the corresponding automorphism θ has GIB.*

Proof. For $a = 0, 1$ the statement follows from Example 4.1 and Proposition 4.5. In cases $a = 2, 3, 4$, GIB was checked according to Algorithm 2.7. Direct verification by hand is also possible. □

Proposition 4.7. *Suppose that $\hat{r} = (2, 2, a)$. Then θ has GIB for all a.*

Proof. Due to Lemma 4.6, the statement is true for $a \leqslant 4$. Assume that $a > 4$. Suppose that $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$ with $\dim \mathbb{V} = a+4$. Let $\mathfrak{h} = \mathfrak{gl}(\mathbb{F}^8)$ with $\mathbb{F}^8 \subset \mathbb{V}$ be a θ-invariant subalgebra of $\mathfrak{g}$ such that the restriction of θ to $\mathfrak{h}$ is an automorphism with $\hat{r}_{\mathfrak{h}} = (2, 2, 4)$. Let also $H \subset G$ be a connected subgroup with $\operatorname{Lie} H = \mathfrak{h}$. Take a nilpotent element $e \in \mathfrak{g}_1$. It must have at least $a - 4$ Jordan blocks of size zero such that $\theta(w_i) = \zeta^2 w_i$. Therefore $G_0{\cdot}e \cap \mathfrak{h}_1 \neq \varnothing$ and we may (and will) assume that $e \in \mathfrak{h}_1$.

Let $\mathfrak{m}$ stand for the $\mathfrak{h}$-invariant complement of $\mathfrak{h}_1$ in $\mathfrak{g}_1$. As a linear space $\mathfrak{m} = M_{a-4,2}(\mathbb{F}) \oplus M_{2,a-4}(\mathbb{F})$. Let $\mathfrak{p}, \mathfrak{p}_- \subset \mathfrak{gl}_a \subset \mathfrak{g}_0$ be two opposite parabolic subalgebras with the Levy part $\mathfrak{gl}_4 \oplus \mathfrak{gl}_{a-4}$ and $\mathfrak{u}_1, \mathfrak{u}_2$ their nilpotent radicals. Note that $[\mathfrak{u}_1, \mathfrak{u}_1] = [\mathfrak{u}_2, \mathfrak{u}_2] = 0$ and $\dim \mathfrak{u}_1 = \dim \mathfrak{u}_2 = 4(a-4)$. Next $\mathfrak{h}_1 = V_1 \oplus V_2 \oplus V_3$, where $V_1 = M_{4,2}(\mathbb{F})$, $V_2 = M_{2,2}(\mathbb{F})$, and $V_3 = M_{2,4}(\mathbb{F})$ are H_0-invariant subspaces. Let us write $v = x_1 + x_2 + x_3$ in accordance with this decomposition of $\mathfrak{h}_1$. We suppose that $\mathfrak{u}_1$ is presented by above-diagonal matrices and $\mathfrak{u}_2$ by below-diagonal. Then $[\mathfrak{u}_1, x_1] = [\mathfrak{u}_2, x_3] = 0$ and $[(\mathfrak{u}_1 \oplus \mathfrak{u}_2), x_2] = 0$. Apart from this we have $[(\mathfrak{u}_1)_{x_3}, w] = M_{2,a-4}(\mathbb{F})$ for generic $w \in V_3$. Similar equality holds for $\mathfrak{u}_2$. As a consequence, for generic $w \in \mathfrak{h}_1$, the subspace $[(\mathfrak{u}_1 \oplus \mathfrak{u}_2)_e, w]$ coincides with $\mathfrak{m}$. Hence

$$[\mathfrak{g}_{0,e}, w] + [\mathfrak{g}_0, e] \supset ([\mathfrak{h}_{0,e}, w] + [\mathfrak{h}_0, e]) \oplus [(\mathfrak{u}_1)_e, w] \oplus [(\mathfrak{u}_2)_e, w] = ([\mathfrak{h}_{0,e}, w] + [\mathfrak{h}_0, e]) \oplus \mathfrak{m}$$

for a generic $w \in \mathfrak{h}_1$. By Lemma 4.6, the pair $(H_0, \mathfrak{h}_1)$ has GIB. Taking the intersection of two open subsets of $\mathfrak{h}_1$, we may assume that w yields also an $H_{0,e}$-orbit of codimension 2 in $\mathfrak{h}_1/[\mathfrak{h}_0, e]$. Then $G_{0,e}$-orbit of w is of codimension at most 2 in $\mathfrak{g}_1/[\mathfrak{g}_0, e]$. Hence the pair $(G_0, \mathfrak{g}_1)$ has GIB as well. □

Example 4.8. Let θ be an automorphism of $\mathfrak{gl}_8$ of order 3 with $\hat{r} = (3, 3, 2)$. Then there is no GIB here. Indeed, take a nilpotent element $e \in \mathfrak{g}_1$ having Jordan blocks $(5, 3)$ with $\theta(w_1) = w_1$ and $\theta(w_2) = \zeta w_2$. We can choose bases of $\mathfrak{g}_{0,e}$ and $\mathfrak{g}_{-1,e}$ as follows:

$$z, \xi_1^{1,0}, \xi_1^{1,3}, \xi_1^{2,2}, \xi_2^{1,4}; \qquad \xi_1^{1,2}, \xi_1^{2,1}, \xi_2^{1,3}, \xi_2^{2,2}.$$

The nilpotent radical of $\mathfrak{g}_{0,e}$ is three dimensional and is generated by the last three basis vectors. By (3.1) it commutes with $\mathfrak{g}_{-1,e}$, for example, $[\xi_1^{2,2}, \xi_2^{1,3}] = \xi_2^{2,5} - \xi_1^{1,5} = 0$. Therefore the matrix $([\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}])$ has rank at most 1 and, due to Lemma 2.6, $\operatorname{ind}(\mathfrak{g}_{0,e}, \mathfrak{g}^*_{-1,e}) \geqslant 3 > 2$.

In the next example, among 191 nilpotent G_0-orbits in $\mathfrak{g}_1$ there are three bad ones, without GIB. One of them is presented here. Others arise after cyclic permutations of θ-eigenvalues on w_1, w_2, w_3.

Example 4.9. Let θ be an automorphism of $\mathfrak{gl}_9$ of order 3 with $\hat{r} = (3,3,3)$. Consider a nilpotent element $e \in \mathfrak{g}_1$ with Jordan blocks of sizes $(5,3,1)$, where w_1 is θ-invariant, $\theta(w_2) = \zeta w_2$, and $\theta(w_3) = \zeta^2 w_3$ (the same as in Example 3.4). The subspaces $\mathfrak{g}_{0,e}$ and $\mathfrak{g}_{-1,e}$ have bases

$$z, \xi_2^{2,0}, \xi_3^{3,0}, \xi_1^{1,3}, \xi_1^{2,2}, \xi_2^{1,4}; \quad \text{and} \quad \xi_1^{1,2}, \xi_1^{2,1}, \xi_1^{3,0}, \xi_2^{1,3}, \xi_2^{2,2}, \xi_3^{1,4},$$

respectively. Using (3.1) it is not difficult to verify that the nilpotent part of $\mathfrak{g}_{0,e}$, generated by $\xi_1^{1,3}, \xi_1^{2,2}, \xi_2^{1,4}$, commutes with $\mathfrak{g}_{-1,e}$. Hence $\operatorname{ind}(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}) \geqslant 6-2 = 4 > 3 = \operatorname{ind}(\mathfrak{g}_0, \mathfrak{g}_1)$.

These examples lead to the following statement.

Proposition 4.10. *Suppose that $\hat{r} = (r_0, r_1, r_2)$ with $r_0, r_1 > 2$ and $r_2 \geqslant 2$. Then $(G_0, \mathfrak{g}_1)$ does not have GIB.*

Proof. Due to Lemma 3.3, it suffices to prove the statement for $\hat{r}$ such that $\hat{r}' = (r_0-1, r_1-1, r_2-1)$ does not satisfy the assumptions after any cyclic permutation. This is possible an exactly two cases: $r_2 = 2$ or if at least two of the numbers r_0, r_1, r_2 are equal to 3.

Suppose first that $\hat{r} = (a, b, 2)$ with $a, b > 2$. Let $e \in \mathfrak{g}_1$ be the same nilpotent element as in Examples 4.8,4.9, i.e., e has Jordan blocks of sizes $(5, 3, 1^{a+b-6})$. For the generators of these Jordan blocks holds: $\theta(w_1) = w_1$, $\theta(w_2) = \zeta w_2$, $\theta(w_i) = w_i$ for $3 \leqslant i \leqslant a-1$, and $\theta(w_j) = \zeta w_j$ for $a \leqslant j \leqslant a+b-2$. Let $\mathbb{F}^8 \subset \mathbb{V}$ be a θ-invariant subspace such that the restriction of θ to $\mathfrak{h} := \mathfrak{gl}(\mathbb{F}^8)$ is defined by $\hat{r}_{\mathfrak{h}} = (3,3,2)$. Let $\mathfrak{f} \cong \mathfrak{gl}_{a+b-6}$ be a θ-invariant subalgebra such that $\mathfrak{h} \oplus \mathfrak{f} \subset \mathfrak{g}$ is a Levi subalgebra of $\mathfrak{g}$. Set $\mathfrak{a} := \mathfrak{h}_{-1,e} = \mathfrak{h} \cap \mathfrak{g}_{-1,e}$. Then $\dim \mathfrak{a} = 4$ and as a vector space $\mathfrak{a}$ is generated by $\xi_1^{1,2}, \xi_1^{2,1}, \xi_2^{1,3}$, and $\xi_2^{2,2}$. Let $\mathfrak{m} \subset \mathfrak{g}_{0,e}$ be a subspace generated by $\xi_i^{j,s} \in \mathfrak{g}_{0,e}$ such that either $i = 1,2$ and $j > 2$ (then necessary $s = 0$), or $i > 2$ and $j = 1,2$ (then necessary $s = d_j$). Then $\mathfrak{g}_{0,e} = \mathfrak{h}_{0,e} \oplus \mathfrak{m} \oplus \mathfrak{f}_0$. Suppose $\eta \in \mathfrak{m}$. Among $\xi_1^{1,t}, \xi_1^{2,t}, \xi_2^{1,t}, \xi_2^{2,t} \in \mathfrak{h}_e$ the element η does not commute only with $\xi_1^{1,0}, \xi_1^{2,0}, \xi_2^{1,2}, \xi_2^{2,0}$. In particular, $[\mathfrak{a}, \eta] = 0$. Note also that $[\mathfrak{a}, \mathfrak{f}] = 0$. Hence $\operatorname{rank}([\mathfrak{g}_{0,e}, \mathfrak{a}]) = \operatorname{rank}([\mathfrak{h}_{0,e}, \mathfrak{a}]) \leqslant 1$ (the inequality was shown to be true in Example 4.8). Due to Lemma 2.6,

$$\operatorname{ind}([\mathfrak{g}_{0,e}, \mathfrak{g}_{-,e}]) = \dim \mathfrak{g}_{-1,e} - \operatorname{rank}([\mathfrak{g}_{0,e}, \mathfrak{g}_{-1.e}]) \geqslant \dim \mathfrak{a} - \operatorname{rank}([\mathfrak{g}_{0,e}, \mathfrak{a}]) \geqslant 3.$$

Therefore $\operatorname{ind}(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}) > \operatorname{ind}(\mathfrak{g}_0, \mathfrak{g}_1) = 2$.

Now we pass to the second case, where $r_2 \geqslant 3$ and at least two of the numbers r_0, r_1, r_2 are equal to 3. Without loss of generality we may assume that $\hat{r} = (3,3,a)$ with $a \geqslant 3$. Here we use the same subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and almost the same nilpotent element $e \in \mathfrak{h}_1$. The only difference is that now e has $a-2$ Jordan blocks of size 1 and for them holds $\theta(w_i) = \zeta^2 w_i$, if $3 \leqslant i \leqslant a$. We have $\mathfrak{g}_{0,e} = \mathfrak{h}_{0,e} \oplus \mathfrak{gl}_{a-2}$ and $\mathfrak{g}_{-1,e} = \mathfrak{h}_{-1,e} \oplus \mathbb{F}^{a-2} \oplus (\mathbb{F}^{a-2})^*$, where $\mathfrak{gl}_{a-2}$ commutes with $\mathfrak{h}_{-1,e}$ and acts on $\mathbb{F}^{a-2}$ via the defining representation. The linear subspace $\mathbb{F}^{a-2} \subset \mathfrak{g}_{-1,e}$ consists of the vectors $\xi_i^{1,4}$ with $i > 2$, while $(\mathbb{F}^{a-2})^* \subset \mathfrak{g}_{-1,e}$ consists of the vectors $\xi_1^{i,0}$ with $i > 2$. Let us choose a basis in $\mathfrak{g}_{0,e}$ such that the first five elements are $z, \xi_2^{2,0}, \xi_1^{1,3}, \xi_1^{2,2}, \xi_2^{1,4}$ and the other $(a-2)^2$ form a basis in $\mathfrak{gl}_{a-2}$. Here

$z, \xi_2^{2,0}, \xi_1^{1,3}, \xi_1^{2,2}, \xi_2^{1,4}$ commute with $\mathbb{F}^{a-2} \oplus (\mathbb{F}^{a-2})^*$. Note also that $\mathfrak{gl}_{a-2} \subset \mathfrak{g}_{0,e}$ commutes with $\mathfrak{h}_{-1,e}$. Hence $\mathrm{rank}([\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}])$ is equal to the sum of $\mathrm{rank}([\mathfrak{h}_{0,e}, \mathfrak{h}_{-1,e}])$ and the rank of the matrix corresponding to the action of $\mathfrak{gl}_{a-2}$ on $\mathbb{F}^{a-2} \oplus (\mathbb{F}^{a-2})^*$, which is $2(a-2)-1$. Summing up, the rank in question is smaller than or equal to $1 + 2(a-2) - 1 = 2(a-2)$. By Lemma 2.6, $\mathrm{ind}\,(\mathfrak{g}_{0,1}, \mathfrak{g}_{-1,e}) \geqslant 4 + 2(a-2) - 2(a-2) = 4 > 3 = \mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1)$. □

Combining Example 4.1 and Propositions 4.5, 4.7, 4.10 we get the following theorem.

Theorem 4.11. *The only inner automorphisms of $\mathfrak{gl}_n$ of order 3, which have the GIB property correspond to the following vectors $\hat{r}$:*

$$(a, b, 0), (a, b, 1), (2, 2, a).$$

The case $m = 3$ is settled now and we pass to higher orders. If $m > 3$, then there are automorphisms of rank 1 without GIB.

Example 4.12. Suppose that $\hat{r} = (2, 2, 2, 1)$. Then there is no GIB. Take a nilpotent element $e \in \mathfrak{g}_1$ with Jordan blocks $(3, 3, 1)$ and $\theta(w_1) = w_1$, $\theta(w_2) = \zeta^2 w_2$, $\theta(w_3) = \zeta w_3$. Computing the stabiliser we get

$$\mathfrak{g}_{-1,e} = \langle \xi_1^{2,1}, \xi_2^{1,1}, \xi_2^{3,0}, \xi_3^{2,2} \rangle_{\mathbb{F}}, \qquad G_{0,e} = (\mathbb{F}^\times)^3 \ltimes \exp\left(\langle \xi_1^{2,2}, \xi_2^{1,2} \rangle_{\mathbb{F}}\right).$$

Note that the nilpotent part of $\mathfrak{g}_{0,e}$, generated by $\xi_1^{2,2}$ and $\xi_2^{1,2}$, acts on $\mathfrak{g}_{-1,e}$ (and hence on its dual) trivially. Therefore $\mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}) \geqslant 2 > \mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1)$.

Example 4.13. Suppose that $\hat{r} = (2, 2, 2, 2)$. Then there is no GIB. Take a nilpotent element $e \in \mathfrak{g}_1$ with Jordan blocks $(3, 3, 1, 1)$ and $\theta(w_1) = w_1$, $\theta(w_2) = \zeta^2 w_2$, $\theta(w_3) = \zeta w_3$, $\theta(w_4) = \zeta^3 w_4$. Computing the stabiliser we get

$$\mathfrak{g}_{-1,e} = \langle \xi_1^{2,1}, \xi_1^{4,0}, \xi_2^{1,1}, \xi_2^{3,0}, \xi_3^{2,2}, \xi_4^{1,2} \rangle_{\mathbb{F}}, \qquad G_{0,e} = (\mathbb{F}^\times)^4 \ltimes \exp\left(\langle \xi_1^{2,2}, \xi_2^{1,2} \rangle_{\mathbb{F}}\right).$$

The nilpotent part of $\mathfrak{g}_{0,e}$, generated by $\xi_1^{2,2}$ and $\xi_2^{1,2}$, acts on $\mathfrak{g}_{-1,e}$ (and hence on its dual) trivially. Therefore $\mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}) \geqslant 3 > \mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1)$.

Theorem 4.14. *Suppose that $m \geqslant 4$ and either $\mathrm{rank}(G_0, \mathfrak{g}_1) > 1$ or $\mathrm{rank}(G_0, \mathfrak{g}_1) = 1$ and $\hat{r}$ contains a subsequence (a, b, c) with $a, b, c \geqslant 2$. Then the corresponding θ-representation has no GIB.*

Proof. Suppose first that $\mathrm{rank}(G_0, \mathfrak{g}_1) > 1$. Passing to a slice representation (as in Lemma 3.3), we may assume that $\mathrm{rank}(G_0, \mathfrak{g}_1) = 2$. Let $e \in \mathfrak{g}_1$ be a nilpotent element with Jordan blocks $(m-1, m-1, 1, \ldots, 1)$ such that $\theta(w_1) = w_1$, $\theta(w_2) = \zeta^{m-2} w_2$, $\theta(w_i) = \zeta w_i$ for $3 \leqslant i \leqslant r_1 + 1$, and $\theta(w_j) = \zeta^{m-1} w_j$ for $r_1 + 2 \leqslant j \leqslant r_1 + r_{m-1}$. The remaining generators w_i with $i > r_1 + r_2$ cannot have eigenvalues ζ or ζ^{m-1}. Let $\mathfrak{h} = \mathfrak{gl}(V)$ be a subalgebra of $\mathfrak{g}$ such that $V \subset \mathbb{V}$ is a θ-invariant subspace of dimension $2m + r_1 + r_{m-1} - 4$, the restriction of θ to $\mathfrak{h}$ is an automorphism defined by $\hat{r}_{\mathfrak{h}} = (2, r_1, 2, \ldots, 2, r_{m-1})$, and, finally, $e \in \mathfrak{h}_1$. Set $a = r_1 - 1$, $b = r_{m-1} - 1$. We have

$$\mathfrak{h}_{-1,e} = \mathbb{F}\xi_1^{2,m-3} \oplus \mathbb{F}\xi_2^{1,1} \oplus (\mathbb{F}^b \oplus (\mathbb{F}^b)^*) \oplus (\mathbb{F}^a \oplus (\mathbb{F}^a)^*),$$

where, for example, $\mathbb{F}^b$ is generated by $\xi_1^{i,0}$ with $a+3 \leqslant i \leqslant r_1+r_{m-1}$. Suppose that $\xi = \xi_i^{j,s} \in \mathfrak{g}_{0,e}$ and $[\xi, \mathfrak{h}_{-1,e}] \neq 0$. Then $s=0$ and either $i=j \in \{1,2\}$ or $\xi \in \mathfrak{gl}_a \oplus \mathfrak{gl}_b \subset \mathfrak{h}_{0,e}$. In any case $\xi \in \mathfrak{h}_{0,e}$. Hence $\mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}) \geqslant \mathrm{ind}\,(\mathfrak{h}_{0,e}, \mathfrak{h}_{-1,e})$. Let $H \subset G$ be a connected subgroup with $\mathrm{Lie}\, H = \mathfrak{h}$. Then

$$H_{0,e} = (\mathbb{F}^\times)^2 \times \mathrm{GL}_a \times \mathrm{GL}_b \ltimes \exp\left(\langle \xi_1^{2,m-2}, \xi_2^{1,2} \rangle_{\mathbb{F}}\right).$$

The nilpotent part of $\mathfrak{h}_{0,e}$ acts on $\mathfrak{h}_{-1,e}$ trivially. Hence $\mathrm{ind}\,(\mathfrak{h}_{0,e}, \mathfrak{h}_{-1,e}) = 3$. Therefore $\mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}) \geqslant 3 > 2 = \mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1)$.

Suppose now that $\mathrm{rank}(G_0, \mathfrak{g}_1) = 1$. Without loss of generality we may assume that $r_0, r_1, r_2 \geqslant 2$. Take $e \in \mathfrak{g}_1$ with Jordan blocks $(m-1, 3, 1, \ldots, 1)$ such that $\theta(w_1) = \zeta^2 w_1$, $\theta(w_2) = w_2$, and $\theta(w_i) = \zeta w_i$ for $3 \leqslant i \leqslant r_1+1$, Let $\mathfrak{a} \subset \mathfrak{g}_{-1,e}$ be a subspace generated by the vectors $\xi_1^{2,1}$, $\xi_2^{1,m-3}$, and $\xi_1^{i,0}, \xi_i^{1,m-2}$ with $3 \leqslant i \leqslant r_1+1$. Then $[\mathfrak{g}_{0,e}, \mathfrak{a}] \subset \mathfrak{a}$ and $G_{0,e}$ acts on $\mathfrak{a}$ as $\mathbb{F}^\times \times \mathrm{GL}_p$ on $\mathbb{F} \oplus \mathbb{F}^* \oplus \mathbb{F}^a \oplus (\mathbb{F}^a)^*$, where $a = r_1 - 1$. Hence $\mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}) \geqslant 2 > 1$. $\square$

Proposition 4.15. *Combining Theorem 4.14 with Lemma 4.4, we see that an automorphism θ of order 6 and rank zero defined by $\hat{r} = (1, a, b, c, 1, 0)$ with $a, b, c \geqslant 2$ has no GIB.*

If $\mathrm{rank}(G_0, \mathfrak{g}_1) = 1$ and 1 occurs often enough among the r_i's, then the θ-representation has GIB. For example, an automorphism with $\hat{r} = (1, 2, 1, 2, 1, 2, \ldots, 1, 2)$ always has GIB. In order to formalise the statement, we need a result in the rank zero case.

Proposition 4.16. *Suppose that $\hat{r} = (1, r_1, \ldots, r_{m-3}, 1, 0)$ has no substrings r_i, r_{i+1}, r_{i+2} with all elements being larger than 1. In other words, if $r_i > 1$, then either r_{i+1} or r_{i+2} must be 1 or 0. Then the corresponding automorphism θ has GIB.*

Proof. We argue by induction on m. In case $m = 3$ the statement is obvious, in cases $m = 4, 5$ it was proved in Lemmas 4.2, 4.3. Suppose that $m > 5$. Let $i \geqslant 1$ be the smallest number such that $r_i \leqslant 1$. By the assumptions $i \leqslant 3$. Set $H = \mathrm{GL}_{r_0} \times \ldots \times \mathrm{GL}_{r_i}$ and $\tilde{H} = \mathrm{GL}_{r_{i+1}} \times \ldots \times \mathrm{GL}_{r_{m-2}}$. Then $G_0 = H \times \tilde{H}$. As usual $\mathfrak{h} = \mathrm{Lie}\, H$ and $\tilde{\mathfrak{h}} = \mathrm{Lie}\, \tilde{H}$. Let $\mathfrak{w} \subset \mathfrak{g}_1$ be the maximal subspace consisting of $\tilde{H}$-invariant vectors and $\tilde{\mathfrak{w}} \subset \mathfrak{g}_1$ its G_0-invariant complement. We have $\mathrm{ind}\,(\mathfrak{h}, \mathfrak{w}) = 0$ and the pair $(H, \mathfrak{w})$ has GIB by the inductive hypothesis. Take $e = x + \tilde{x}$ with $x \in \mathfrak{w}$, $\tilde{x} \in \tilde{\mathfrak{w}}$. According to Lemma 2.3, there is $y \in \mathfrak{w}$ such that $[\mathfrak{h}_x, w] + [\mathfrak{h}, x] = \mathfrak{w}$. Let $T, \tilde{T}$ be the central tori of H and $\tilde{H}$, respectively, and set $\mathfrak{t} = \mathrm{Lie}\, T$, $\tilde{\mathfrak{t}} = \mathrm{Lie}\, \tilde{T}$. Note that T acts trivially on $\mathfrak{w}$. In particular, $T \subset (H_x)_y$. The action of $T \times \tilde{H}$ on $\tilde{\mathfrak{w}}$ is a θ-representation corresponding to a vector $(r_i, r_{i+1}, \ldots, r_{m-3}, 1, 0)$. Therefore it has GIB by the inductive hypothesis and there is $\tilde{y} \in \tilde{\mathfrak{w}}$ such that $[(\mathfrak{t} \oplus \tilde{\mathfrak{h}})_{\tilde{x}}, \tilde{y}] + [\mathfrak{t} \oplus \tilde{\mathfrak{h}}, \tilde{x}] = \tilde{\mathfrak{w}}$. Combining these two equalities we get

$$\begin{aligned}[]
[\mathfrak{g}_0, e] + [\mathfrak{g}_{0,e}, y+\tilde{y}] &= [\mathfrak{g}_0, e] + [\mathfrak{t} \oplus \tilde{\mathfrak{h}}, \tilde{x}] + [\mathfrak{g}_{0,e}, y+\tilde{y}] + [(\mathfrak{t} \oplus \tilde{\mathfrak{h}})_{\tilde{x}}, \tilde{y}] = \\
&= \tilde{\mathfrak{w}} + [\mathfrak{h}, x] + [\mathfrak{g}_{0,e}, y] = \tilde{\mathfrak{w}} + [\mathfrak{h}, x] + [\mathfrak{g}_{0,e} + \tilde{\mathfrak{h}}, y].
\end{aligned}$$

It remains to understand $[\mathfrak{g}_{0,e} + \mathfrak{h}, y]$. We claim that $\mathfrak{h}_x \subset \mathfrak{g}_{0,e} + \tilde{\mathfrak{h}}$. In case $\mathfrak{h}_x$ acts on $\tilde{\mathfrak{w}}$ trivially, the claim is obvious ($\mathfrak{h}_x \subset \mathfrak{h}_e \subset \mathfrak{g}_{0,e}$). If the action is not trivial, then H_x acts on $\tilde{\mathfrak{w}}$ as GL_{r_i} and $[\mathfrak{h}_x, y] = [\tilde{\mathfrak{t}}, y]$. Therefore $\mathfrak{h}_x \subset (\mathfrak{h}_x \oplus \tilde{\mathfrak{t}})_y + \tilde{\mathfrak{t}} \subset (\mathfrak{g}_{0,x})_y + \tilde{\mathfrak{t}}$. The claim is proved.

The inclusion $\mathfrak{h}_x \subset \mathfrak{g}_{0,e} + \tilde{\mathfrak{h}}$ implies that $[\mathfrak{g}_{0,e} + \tilde{\mathfrak{h}}, y] + [\mathfrak{h}, x] \supset [\mathfrak{h}_x, y] + [\mathfrak{h}, x] = \mathfrak{w}$. We see that $[\mathfrak{g}_0, e] + [\mathfrak{g}_{0,e}, y+\tilde{y}] = \mathfrak{w} \oplus \tilde{\mathfrak{w}} = \mathfrak{g}_1$ and hence $(G_0, \mathfrak{g}_1)$ has GIB by Lemma 2.3 . $\square$

Combining Proposition 4.16 with Lemma 4.4, we get the following.

Proposition 4.17. *Suppose that $\hat{r} = (1, r_1, \ldots, r_{m-2}, r_{m-1})$ has no substrings r_i, r_{i+1}, r_{i+2} with all elements being larger than 1. Then the corresponding automorphism θ has GIB.*

The rank zero case we leave open. Mainly due to the following observations.

Lemma 4.18. *Let q and n be natural numbers such that $2q \leqslant n$. Then the representation of $\mathfrak{gl}_n$ on $V = (q\mathbb{F}^n)^*$, and hence on V^*, has GIB.*

Proof. Since q is smaller than n, the GL_n-orbits on V are classified by the matrix rank. Let $v \in V$ be a matrix of rank p ($p \leqslant q$). In case $p = q$ the orbit is of the maximal dimension and it satisfies GIB. Assume $p < q$. The quotient space $V/\mathfrak{gl}_n{\cdot}v$ is isomorphic to $q\mathbb{F}^{n-p}$ and $(\mathrm{GL}_n)_v$ acts on it as GL_{n-p}. We have $q < (n-p)$, because $q + p < 2q \leqslant n$. Hence there is an open $(\mathrm{GL}_n)_v$-orbit in the quotient $V/\mathfrak{gl}_n{\cdot}v$. $\square$

Lemma 4.18 leads to the following propagation property.

Proposition 4.19. *Suppose that a zero rank automorphism θ of $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$ with $\hat{r} = (0, r_1, r_2, \ldots, r_{m-1})$ has GIB and $q \geqslant 2r_{m-1}$. Then an automorphism of $\mathfrak{h} = \mathfrak{gl}(\mathbb{V} \oplus \mathbb{F}^q)$ of order $m+1$ defined by $\hat{r}' := (0, r_1, r_2, \ldots, r_{m-1}, q)$ has GIB as well.*

Proof. We have $\mathfrak{h}_0 = \mathfrak{g}_0 \oplus \mathfrak{gl}_q$ and $\mathfrak{h}_1 = \mathfrak{g}_1 \oplus V$, where $V = (\mathbb{F}^{r_{m-1}})^* \otimes \mathbb{F}^q$. Take $x \in \mathfrak{h}_1$. It decomposes as $x = e + v$, where $e \in \mathfrak{g}_1$ and $v \in V$. Clearly $(\mathfrak{gl}_q)_v \subset \mathfrak{h}_{0,x}$. Because $q \geqslant r_{m-1}$, we have also $\mathfrak{g}_{0,e} \subset \mathfrak{h}_{0,x} + \mathfrak{gl}_q$. Both pairs $(G_0, \mathfrak{g}_1)$ and (GL_q, V^*) have GIB. Due to Lemma 2.3, there are $\xi \in \mathfrak{g}_1$, $w \in V$ such that $[\mathfrak{g}_{0,e}, \xi] + [\mathfrak{g}_0, e] = \mathfrak{g}_1$ and $(\mathfrak{gl}_q)_v{\cdot}w + \mathfrak{gl}_q{\cdot}v = V$. Set $y = \xi + w$. Then

$$\begin{aligned}[][\mathfrak{h}_0, x] + [\mathfrak{h}_{0,x}, y] &= [\mathfrak{gl}_q, v] + [(\mathfrak{gl}_q)_v, w] + [\mathfrak{g}_0, x] + [\mathfrak{g}_{0,x}, y] = V + [\mathfrak{g}_0, e] + [\mathfrak{g}_{0,x}, \xi] = \\ &= V + [\mathfrak{g}_0, e] + [(\mathfrak{g}_{0,x} + \mathfrak{gl}_q), \xi] \supset V + [\mathfrak{g}_0, e] + [\mathfrak{g}_{0,e}, \xi] = \mathfrak{h}_1.\end{aligned}$$

Thereby $(H, \mathfrak{h}_1)$ has GIB by Lemma 2.3. $\square$

It seems that in the rank zero case the GIB property depends on the entries r_i in a rather bizarre way. However computations in small dimensions indicate that most likely GIB holds if the order of θ is 4 or 5.

Conjecture 4.20. The GIB property holds for rank zero inner automorphisms of $\mathfrak{gl}_n$ of orders 4 and 5.

Remark 4.21. Semisimple automorphisms of infinite order yield exactly the same θ-representations as finite order automorphisms of rank zero. Therefore the problem of checking GIB also remains open for infinite order automorphisms.

5. GIB IN THE EXCEPTIONAL TYPES

We have implemented Algorithm 2.7 in GAP4, using the functionality for listing nilpotent orbits of θ-groups present in the SLA package ([6]). For computing the rank of $\widetilde{A}(\underline{a})$, where the a_i are indeterminates of a function field, we have used MAGMA ([1]).

We have used this implementation to find automorphisms θ of the Lie algebras of exceptional type for which $\mathrm{rank}(G_0, \mathfrak{g}_1) > 0$ and $(G_0, \mathfrak{g}_1)$ has GIB.

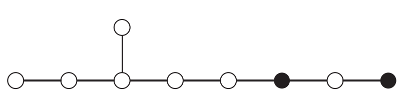
In Tables 1 to 6 the Kac diagrams of the automorphisms that we found to have GIB are listed. The explanation of the content of these tables is as follows. Since we restrict ourselves to automorphisms of positive rank, the labels of the Kac diagrams are 0,1. Hence we give these labels by colouring the nodes in the Kac diagram: black means that the label is 1, otherwise the label is 0. Note that we can restrict to automorphisms of order less then the Coxeter number. Indeed, if the order is equal to that number, then G_0 will be a torus, and hence $(G_0, \mathfrak{g}_1)$ has GIB by [13, Proposition 1.3]. For higher orders $(G_0, \mathfrak{g}_1)$ has rank zero. In the tables, the first column has the order of θ, and the second column its Kac diagram. The third column has the rank of $(G_0, \mathfrak{g}_1)$. Moreover, in order to save space, we put two sets of columns next to each other.

Remark 5.1. Tables 1 and 3 (inner automorphisms of respectively, $\mathbf{E}_6$ and $\mathbf{E}_7$) contain only one automorphism of order 2. In [13] it was left open whether or not these cases have GIB. We conclude that they do.

In all cases where it was necessary to compute the rank of a matrix $\widetilde{A}(\underline{a})$, with a_i indeterminates in a function field, this proved to be a straightforward calculation, except for two cases, both in $\mathbf{E}_8$. In those cases Algorithm 2.7 establishes that GIB does not hold with high probability. Furthermore, exactly one nilpotent orbit is found that very probably causes GIB to fail. However, it proved to be a too demanding calculation to compute the rank of $\widetilde{A}(\underline{a})$, with a_i indeterminates in a function field. These two cases are examined in detail in Examples 5.3, 5.4, where we show that they do not have GIB. We conclude that the following theorem holds.

Theorem 5.2. *Tables 1 to 6 contain the Kac diagrams of all positive rank automorphisms of the Lie algebras of exceptional type, such that $(G_0, \mathfrak{g}_1)$ has GIB.*

Example 5.3. Let θ be an automorphism of $\mathbf{E}_8$ with the following Kac diagram:



Then $(G_0, \mathfrak{g}_1)$ does not have GIB. By the computations made with Algorithm 2.7, there is a unique orbit that is possibly bad. Here $G_0 = E_6 \times \mathrm{SL}_2 \times T$ with $T \cong \mathbb{F}^\times$ and $\mathfrak{g}_1 = V_1 \oplus V_2$ with $V_1 = \mathbb{F}^{27} \otimes \mathbb{F}^2$, $V_2 = \mathbb{F}^2$. The representation of the group E_6 on $\mathbb{F}^{27}$ is of index 1 and for generic $v \in \mathbb{F}^{27}$ the stabiliser $(E_6)_v$ is reductive and of type F_4, see e.g. [3]. Let $v \in \mathbb{F}^{27}$ be such that $(E_6)_v = F_4$. As is very well known, F_4 is the subgroup of stable points for the diagram automorphism of E_6. Hence its normaliser in E_6 coincides with F_4 up to a connected component. Therefore $\mathbb{F}v$ is not contained in the tangent space $[\mathbf{E}_6, v]$. This implies that $T \times E_6$ acts on $\mathbb{F}^{27}$ with an open orbit.

Let $t \in T$ and $v_1 \in V_1$, $v_2 \in V_2$. Then $t{\cdot}(v_1 + v_2) = tv_1 + t^{-3}v_2$. Two copies of $\mathbb{F}^2$ are canonically isomorphic as SL_2-modules. Using this isomorphism we write a representative of the bad orbit as $e = v{\otimes}w + w$, where $w \in \mathbb{F}^2$ and $v \in \mathbb{F}^{27}$ is generic, i.e., such that $(E_6)_v = F_4$.

As we already know, there are no elements $\xi \in \mathbf{E}_6$ such that $\xi{\cdot}v \neq 0$ and $\xi{\cdot}v \in \mathbb{F}v$. Therefore $\mathfrak{g}_{0,v\otimes w} = \mathbf{F}_4 \oplus \mathfrak{h} \oplus \mathfrak{u}$, where $\mathfrak{u} \subset \mathfrak{sl}_2$ is the Lie algebra of a unipotent subgroup, $\mathfrak{h} = \mathbb{F}$, and $\mathfrak{h}$ is embedded diagonally into $\mathfrak{t} \oplus \mathfrak{sl}_2$ (here $\mathfrak{t} = \mathrm{Lie}\,T$). Note that for $h \in \mathfrak{h}$ and $w \in V_2$ being the same as above, we have $h{\cdot}w = -4hw$. Hence $\mathfrak{g}_{0,e} = \mathbf{F}_4 \oplus \mathfrak{u}$.

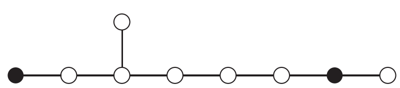
The tangent space $[\mathfrak{g}_0, e]$ is equal to

$$[\mathbf{E}_6, v]{\otimes}w \oplus \mathbb{F}(v{\otimes}w' + w') \oplus \mathbb{F}v{\otimes}w \oplus \mathbb{F}w,$$

where $w' \in \mathbb{F}^2$ is a vector non-proportional to w. Hence $\mathfrak{g}_1/[\mathfrak{g}_0, e] \cong \mathbb{F}^{26}{\otimes}w' \oplus \mathbb{F}$, where the line $\mathbb{F}$ is an anti-diagonal in $\mathbb{F}(v{\otimes}w') \oplus \mathbb{F}w'$. For the nilpotent part $\mathfrak{u} \subset \mathfrak{sl}_2$ of $\mathfrak{g}_{0,e}$, holds $\mathfrak{u}{\cdot}w' = w$. Therefore it acts on $\mathfrak{g}_1/[\mathfrak{g}_0, e]$ trivially. The representation of F_4 on the space $\mathbb{F}^{26}{\otimes}w' \oplus \mathbb{F}$ is the sum of the trivial representation and the simplest one, which has index two. Thereby $\mathrm{ind}\,(\mathfrak{g}_{0,e}, \mathfrak{g}_{-1,e}) = 3$.

There are many ways to see that $\mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1) = 2$. One of them is to take a slightly modified element in $\mathfrak{g}_1$, namely $x = v{\otimes}w + w'$ with w and w' being linear independent. Then $G_{0,x}$-action on $\mathfrak{g}_1/[\mathfrak{g}_0, x]$ is the same as the action of F_4 on $\mathbb{F}^{26}$ (there is no additional line $\mathbb{F}$ here). We have $\mathrm{ind}\,(\mathfrak{g}_{0,x}, \mathfrak{g}_{-1,x}) = 2$ and since the stabiliser $G_{0,x}$ is reductive, $\mathrm{ind}\,(\mathfrak{g}_{0,x}, \mathfrak{g}_{-1,x}) = \mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1)$, see [13, Proposition 1.1].

Example 5.4. Let θ be an automorphism of $\mathbf{E}_8$ corresponding to the following Kac diagram:



Then $(G_0, \mathfrak{g}_1)$ does not have GIB. Here the order of θ is 4, $G_0 = \mathrm{Spin}_{12}{\times}\mathrm{SL}_2{\times}\mathbb{F}^{\times}$, and $\mathrm{rank}(G_0, \mathfrak{g}_1) = 2$. A suspicious orbit $G_0{\cdot}e \subset \mathfrak{g}_1$ was found in accordance with Algorithm 2.7. It has dimension 29 and if we include e into an $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in \mathfrak{g}_0$, $f \in \mathfrak{g}_{-1}$, then $h \in \mathfrak{so}_{12}$ and the characteristic h acts in the defining representation $\mathbb{F}^{12}$ of $\mathfrak{so}_{12}$ as a semisimple matrix with eigenvalues $(2, 2, -2, -2, 0^8)$. Let $\mathfrak{t} \subset \mathfrak{so}_{12}$ be a maximal torus containing h. We also fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{so}_{12}$ containing $\mathfrak{t}$. Replacing h by a G_0-conjugate element, if necessary, we may (and will) assume that $\varepsilon_1(h) = \varepsilon_2(h) = 2$, $\varepsilon_3(h) = \varepsilon_4(h) = \varepsilon_5(h) = \varepsilon_6(h) = 0$ for the standard basis $\{\varepsilon_1, \ldots, \varepsilon_6\}$ of $\mathfrak{t}^*$.

We have a G_0-invariant decomposition $\mathfrak{g}_1 = V \oplus W$, where the semisimple part of G_0 acts on $V = \mathbb{F}^{32}$ via a half-spin representation of Spin_{12}, and on $W = \mathbb{F}^{12}{\otimes}\mathbb{F}^2$ via the tensor product of the defining representations. The semisimple element h is invariant under the diagram automorphism of D_6. Therefore the picture would not change if we replace one half-spin representation by another (this would be just a different choice of the simple roots for $\mathfrak{so}_{12}$). It is more convenient to assume that the highest weight λ of V is equal to $(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_6)/2$. The other weights of V are $(\sum\limits_{i=1}^{6} \pm\varepsilon_i)/2$ with even number of minus signs and each weight space is one-dimensional. Let $v_\lambda \in V$ be a highest weight vector.

In order to identify e in terms of V and W, we need to understand the subspace $\mathfrak{g}_1(2)$, where 2 stands for the eigenvalue of $\mathrm{ad}\,(h)$. Under the action of $G_{0,h} = \mathrm{Spin}_8 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ the subspaces V and W decompose as $V = V_1 \oplus V_2 \oplus V_3$ and $W = W_1 \oplus W_2 \oplus W_3$, where

$V_1 \cong V_3 \cong \mathbb{F}^8_+$ with $\mathbb{F}^8_+$ having the highest weight $(\varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6)/2$;
$V_2 \cong \mathbb{F}^2 \otimes \mathbb{F}^8_-$ with $\mathbb{F}^8_-$ having the highest weight $(\varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6)/2$;
$W_1 \cong W_3 \cong \mathbb{F}^2 \otimes \mathbb{F}^2$, $W_2 \cong \mathbb{F}^8 \otimes \mathbb{F}^2$ with $\mathbb{F}^8$ having the highest weight ε_3.

We assume that v_λ is a highest weight vector in V_1 (not in V_3). Note that $(\sum\limits_{i=1}^{6} \pm\varepsilon_i)(h) = 4$ if and only if ε_1 and ε_2 are taken with the sign $+$. Therefore $\mathfrak{g}_1(2) = V_1 \oplus W_1$ and the group $G_{0,h}$ indeed acts on it with an open orbit. (This proves the existence of a nilpotent element $e \in \mathfrak{g}_1$ with the characteristic h.) As a representative of the open $G_{0,h}$-orbit we choose $v + w$, where $v \in V_1$ is the sum of v_λ and a lowest weight vector $v_\mu \in V_1$ (with respect to Spin_8), i.e., $\mu = (\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6)/2$ as a weight of Spin_{12}; and $w \in W_1 (\cong M_{2,2}(\mathbb{F}^2))$ is the identity matrix.

Let $L \cong \mathrm{GL}_2$ be a normal subgroup of G_0. By a direct computation we get $G_{0,w} = \mathrm{GL}_2 \times \mathrm{Spin}_8 \ltimes N$, where N is the unipotent radical of a parabolic subgroup $P \subset \mathrm{Spin}_{12}$ with the Levi part of type $A_1 \times D_4$ and $\mathfrak{b} \subset \mathrm{Lie}\,P$; and the GL_2-factor is embedded diagonally into $P \times L$. We claim that $N{\cdot}v = v$ or, what is the same, that $[\mathfrak{n}, v] = 0$ for $\mathfrak{n} = \mathrm{Lie}\,N$. Because v_λ is the highest weight vector, and $\mathfrak{n}$ is contained in the nilpotent radical of $\mathfrak{b}$, we have $[\mathfrak{n}, v_\lambda] = 0$. The Lie algebra $\mathfrak{n}$ consists of weight-spaces with weights $\varepsilon_1 + \varepsilon_2$ and $\varepsilon_1 \pm \varepsilon_j$, $\varepsilon_2 \pm \varepsilon_j$ with $3 \leqslant j \leqslant 6$. Clearly each weight of $[\mathfrak{n}, v_\mu]$ has a coefficient $3/2$ in front of ε_1 or ε_2. Hence $[\mathfrak{n}, v_\mu] = 0$ and $[\mathfrak{n}, v] = 0$. The stabiliser of v in the reductive part of $G_{0,w}$, i.e., in $\mathrm{GL}_2 \times \mathrm{Spin}_8$, is equal to $\mathrm{SL}_2 \times \mathrm{Spin}_7$. Therefore $G_{0,e} = \mathrm{SL}_2 \times \mathrm{Spin}_7 \ltimes N$.

Our final goal is to understand the quotient space $\mathfrak{g}_1/[\mathfrak{g}_0, e]$. Let N_- be the unipotent radical of an opposite to P parabolic. Set $\mathfrak{n}_- := \mathrm{Lie}\,N_-$, $\mathfrak{l} := \mathrm{Lie}\,L$. Then $\mathfrak{g}_0 = \mathfrak{g}_{0,w} \oplus \mathfrak{n}_- \oplus \mathfrak{l}$ and

$$[\mathfrak{g}_0, e] = [\mathfrak{g}_{0,w}, v] \oplus [\mathfrak{n}_-, e] \oplus [\mathfrak{l}, w] = V_1 \oplus [\mathfrak{n}_-, e] \oplus W_1.$$

Note that $[\mathfrak{n}_-, e] = \mathbb{F}^{16} \oplus \mathbb{F}$, where $\mathbb{F}^{16}$ is embedded diagonally into $V_2 \oplus W_2$, and $\mathbb{F}$ is embedded diagonally into $V_3^{\mathrm{Spin}_7} \oplus W_3^{\mathrm{SL}_2}$. Therefore $\mathfrak{g}_1/[\mathfrak{g}_0, e] = (\mathbb{F} \oplus \mathbb{F}^3 \oplus \mathbb{F}^7) \oplus \mathbb{F}^8 \otimes \mathbb{F}^2$ as an $(\mathrm{SL}_2 \times \mathrm{Spin}_7)$-module. Here SL_2 acts on $\mathbb{F}^3$ via the adjoint representation. Moreover, in the quotient $\mathfrak{n}{\cdot}(\mathbb{F} \oplus \mathbb{F}^3 \oplus \mathbb{F}^7) \subset \mathbb{F}^8 \otimes \mathbb{F}^2$ and $\mathfrak{n}{\cdot}(\mathbb{F}^8 \otimes \mathbb{F}^2) = 0$. This implies that $\mathrm{ind}\,(\mathfrak{g}_{0,e}, (\mathfrak{g}_1/[\mathfrak{g}_0, e])^*) \geqslant \mathrm{ind}\,(\mathfrak{so}_7 \oplus \mathfrak{sl}_2, \mathbb{F} \oplus \mathbb{F}^3 \oplus \mathbb{F}^7) = 3$. Since $\mathrm{ind}\,(\mathfrak{g}_0, \mathfrak{g}_1) = 2$, we conclude that this pair $(G_0, \mathfrak{g}_1)$ does not have GIB.

Table 1: Inner automorphisms of $\mathbf{E}_6$ such that $(G_0, \mathfrak{g}_1)$ has GIB.

$\lvert\theta\rvert$	Kac diagram	rk	$\lvert\theta\rvert$	Kac diagram	rk

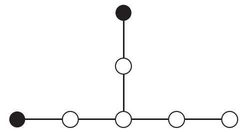
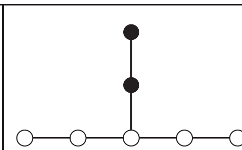
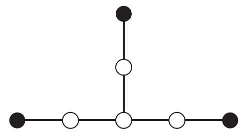
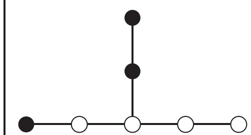
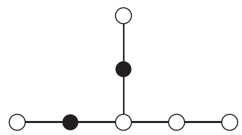
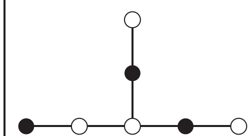
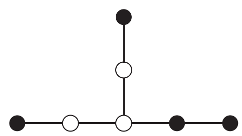
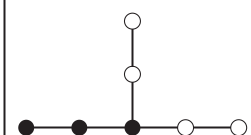
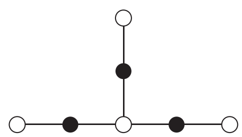
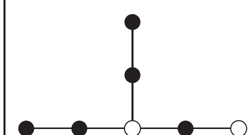
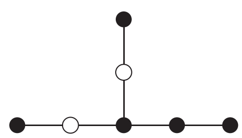
GIB automorphisms of $\mathbf{E}_6$.					
2		2	3		1
3		2	4		1
4		1	5		1
5		1	6		1
6		1	8		1
8		1			

Table 2: Outer automorphisms of $\mathbf{E}_6$ such that $(G_0, \mathfrak{g}_1)$ has GIB.

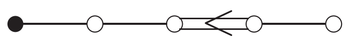
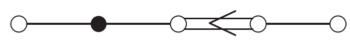
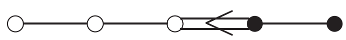
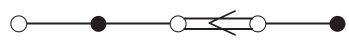
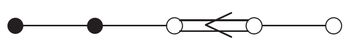
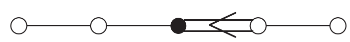
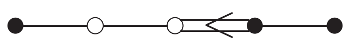
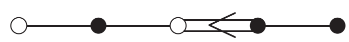
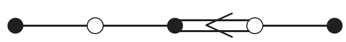
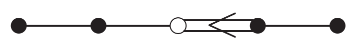
$\lvert\theta\rvert$	Kac diagram	rk	$\lvert\theta\rvert$	Kac diagram	rk
2		2	4		1
6		1	6		2
6		1	6		1
8		1	10		1
10		1	12		1

Table 3: Inner automorphisms of $\mathbf{E}_7$ such that $(G_0, \mathfrak{g}_1)$ has GIB.

$\lvert\theta\rvert$	Kac diagram	rk	$\lvert\theta\rvert$	Kac diagram	rk

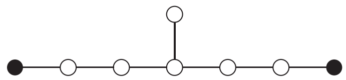
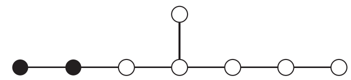
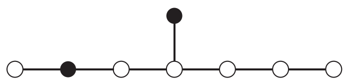
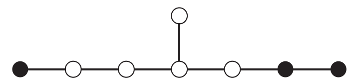
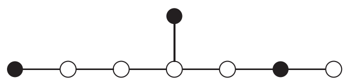
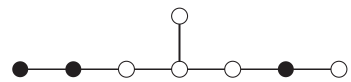
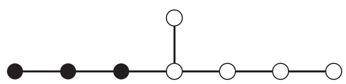
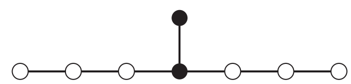
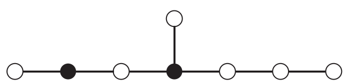
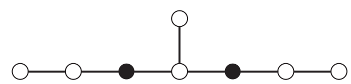
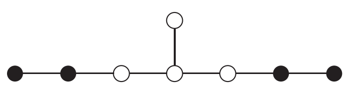
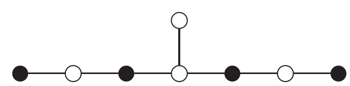
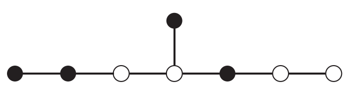
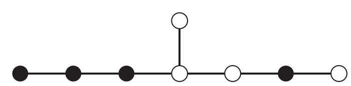
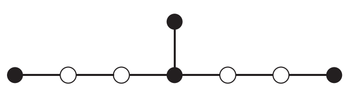
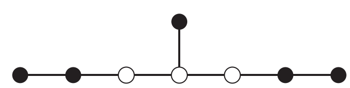
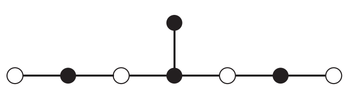
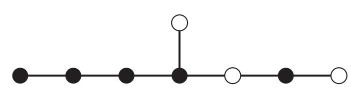
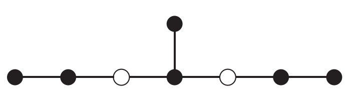
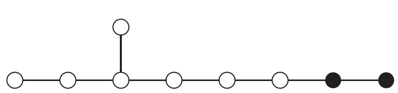
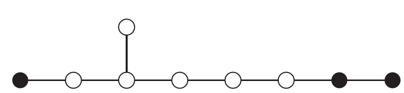
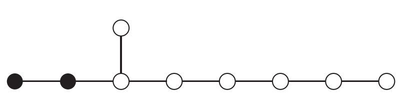
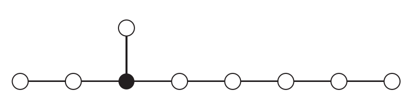
GIB automorphisms of $\mathbf{E}_7$.					
2		3	3		1
4		1	4		1
5		1	5		1
6		1	6		1
6		1	6		1
6		1	8		1
8		1	8		1
8		1	8		1
10		1	12		1
12		1			

Table 4: Automorphisms of $\mathbf{E}_8$ such that $(G_0, \mathfrak{g}_1)$ has GIB.

$\lvert\theta\rvert$	Kac diagram	rk	$\lvert\theta\rvert$	Kac diagram	rk
3		1	5		1
6		1	6		1

GIB automorphisms of $\mathbf{E}_8$.

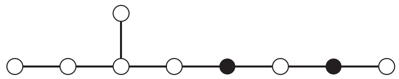
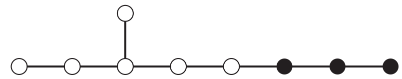
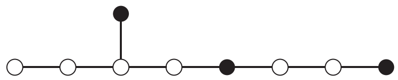
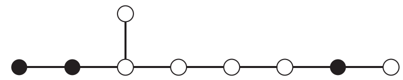
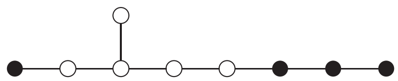
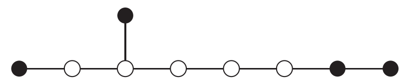
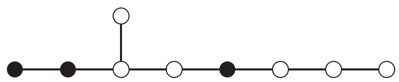
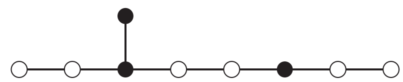
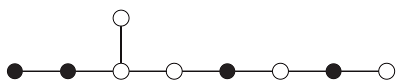
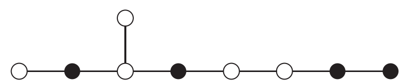
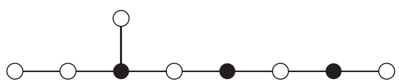
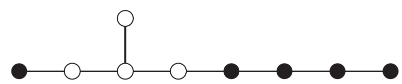
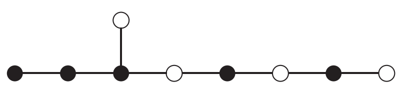
6		1	6		1
8		1	8		1
8		1	8		1
10		1	12		1
12		1	12		1
12		1	12		1
18		1			

Table 5: Automorphisms of $\mathbf{F}_4$ such that $(G_0, \mathfrak{g}_1)$ has GIB.

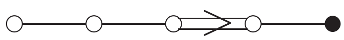
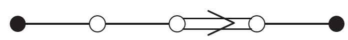
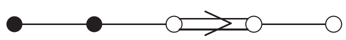
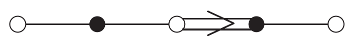
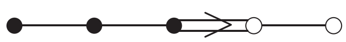
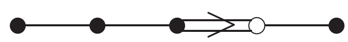
$\lvert\theta\rvert$	Kac diagram	rk	$\lvert\theta\rvert$	Kac diagram	rk
2		1	3		1
3		1	6		1
6		1	8		1

Table 6: Automorphisms of $\mathbf{G}_2$ such that $(G_0, \mathfrak{g}_1)$ has GIB.

$\lvert\theta\rvert$	Kac diagram	rk
3		1

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